Exact Algorithms for Graph Homomorphisms*

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Abstract

Graph homomorphism (also called H-coloring) is a nice generalization of graph coloring: There is a homomorphism from a graph G to a complete graph on k vertices if and only if G is k-colorable. In recent years the topic of exact (exponential-time) algorithms for NP hard problems, and for graph coloring in particular, has led to much research. So it is natural to ask how the techniques developed for exact graph coloring algorithms can be extended to graph homomorphisms. By the celebrated result of Hell & Nešetřil, for each fixed simple graph H, to decide whether a given simple graph G has a homomorphism to H is polynomial time solvable if H is a bipartite graph, and NP-complete otherwise. The case $H = C_5$ (cycle of length 5) is the first NP hard case different from graph coloring. We show that for a given graph G on n vertices and an odd integer $k \geq 5$, the question if G is homomorphic to a cycle of length k can be solved in time $\mathcal{O}(\min\{\binom{n}{n/k}, 2^{n/2}\})$ (up to polynomial factors). We extend the results obtained for cycles, which are graphs of treewidth two, to graphs of bounded treewidth as follows: if the graph H is of treewidth at most ℓ, then the question whether G is homomorphic to H is solvable in time $\mathcal{O}((2\ell+1)^n)$ (up to polynomial factors).

Topic: Design and Analysis of Algorithms.

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1 Introduction

Given two undirected graphs G and H, a *homomorphism* from G to H is a mapping $\varphi : V(G) \longrightarrow V(H)$ that satisfies the following: $\{x, y\} \in E(G) \Longrightarrow \{\varphi(x), \varphi(y)\} \in E(H)$ for every $x, y \in V(G)$. When there is a homomorphism from G to H we say that G is *homomorphic* to H. The problem of deciding whether graph G is homomorphic to graph H is called $\mathrm{HOM}(G, H)$. This problem can be seen as labelling (or coloring) the vertices of G by the vertices of H, that is why it is also often called H-coloring problem. Note that for the special case when H is a complete graph on k vertices, G is homomorphic to H if and only if the chromatic number of G is at most k. We refer to the recent book [13] for a nice introduction to the topic.

For graph classes $\mathcal{G}$ and $\mathcal{H}$ we denote by $\mathrm{HOM}(\mathcal{G}, \mathcal{H})$ the restriction of the graph homomorphism problem to input graphs $G \in \mathcal{G}$ and $H \in \mathcal{H}$. If $\mathcal{G}$ or $\mathcal{H}$ is the class of all graphs then they are denoted by the placeholder '_'. The computational complexity of homomorphism was studied from different 'sides'.

'Left side' of homomorphisms. For any fixed graph G, $\mathrm{HOM}(G, _)$ is trivially solvable in polynomial time. Several authors independently showed that $\mathrm{HOM}(\mathcal{G}, _)$ is solvable in polynomial time if all graphs in $\mathcal{G}$ have bounded treewidth. In this case polynomial time algorithms can be obtained even for counting homomorphisms [8]. Grohe, concluding from results of Dalmau et al. [7], showed that $\mathrm{HOM}(\mathcal{G}, _)$ is solvable in polynomial time if and only if the cores of all graphs in $\mathcal{G}$ have bounded treewidth (under some parameterized complexity theoretic assumptions) [11].

'Right side' of homomorphisms. Hell & Nešetřil showed that for any fixed simple graph H, the problem $\mathrm{HOM}(_, H)$ is solvable in polynomial time if H is bipartite and NP-complete if H is not bipartite [12]. This resolves the complexity classification of the whole right side of homomorphisms, and provides P vs. NP dichotomy. Consequently the study of the right side of homomorphisms (for undirected graphs) almost stopped, as research has been mainly concentrated on finding polynomial time algorithms for special graph classes from the 'left' side.

However for the special case of graph homomorphism, graph coloring, much work has been done recently resulting in faster and faster exponential time algorithms. The recent champions are $\mathcal{O}^*(1.3289^n)$-time algorithm for 3-coloring [4], for 4-coloring $\mathcal{O}^*(1.7504^n)$ [5], for 5-coloring $\mathcal{O}^*(2.1020^n)$ [6] and for 6-coloring $\mathcal{O}^*(2.1809^n)$ time algorithm [6]. For $k \geq 7$, the k-coloring problem can be solved in time $\mathcal{O}^*(2.4023^n)$ [5]. (Throughout this paper we use a modified big-Oh notation that suppresses all polynomially bounded factors. For functions f and g we write $f(n) = \mathcal{O}^*(g(n))$ if $f(n) = g(n) \cdot n^{\mathcal{O}(1)}$.)

Despite much progress on exponential-time algorithms for graph coloring problems, not much is known on exponential-time algorithms for the graph homomorphism problem. By the result of Hell & Nešetřil, $\mathrm{HOM}(_, H)$ is polynomial time solvable when H is bipartite. Another 'easy' case is when $\chi(H) = \omega(H)$, i.e. the chromatic number of H is equal to the maximum clique size. It is not hard to show that in this case the $\mathrm{HOM}(_, H)$ problem is equivalent to the k-coloring problem with

$k = \chi(H)$. Consequently the HOM($_, H$) problem is equivalent to the $\chi(H)$-coloring problem for all perfect graphs H.

All this motivates us to study exact exponential time algorithms for HOM($_, H$) and graphs H with $\chi(H) > \omega(H)$; thus odd chordless cycles are the first natural candidates to study exponential algorithms for graph homomorphisms. For the cycle C_3 on 3 vertices HOM($_, C_3$) is equivalent to 3-coloring but already for the cycle on 5 vertices C_5 no better deterministic algorithm than the brute-force $\mathcal{O}^*(5^n)$ time algorithm was known.

Our results. In this paper we initiate the study of exponential time complexity of homomorphism problems beyond graph coloring. We show that for a graph G on n vertices and an odd integer $k \geq 5$, HOM(G, C_k) is $\mathcal{O}^*(\min\{\binom{n}{n/k}, 2^{n/2}\})$ time solvable, where C_k is the cycle on k vertices. In particular, the running time of our algorithm is $\mathcal{O}^*(1.64939^n)$ when $k = 5$, $\mathcal{O}^*(1.50700^n)$ when $k = 7$, and $\mathcal{O}^*(1.41742^n)$ when $k = 9$, and $\mathcal{O}^*(\alpha^n)$ with $\alpha < \sqrt{2}$ for all $k \geq 11$. It is interesting to note that for $k \geq 15$, our algorithm for homomorphism to C_k is faster than the fastest known 3-coloring algorithm. Hence the natural conjecture that HOM($_, C_k$) is at least as difficult as 3-coloring for every odd $k \geq 5$ might be mistaken. Our algorithms use 2-SAT expressions to search for suitable extensions of an initial partial homomorphism: a maximal independent set of G to be mapped to a carefully chosen subset of vertices of H. To enumerate all possible preliminary choices we use known algorithms to enumerate all maximal independent sets.

Treewidth and tree decompositions are of great importance in structural graph theory and graph algorithms. Many NP hard problems become polynomial or even linear time solvable when the input is restricted to graphs of bounded treewidth. We refer to [3] for a survey on this parameter. It seems that the treewidth can be a useful tool to design exponential time algorithms as well. We use dynamic programming techniques similar to bounded treewidth techniques to solve HOM(G, H) in time $\mathcal{O}^*((2 \cdot \mathbf{tw}(H) + 1)^{|V(G)|})$ assuming an optimal tree decomposition of H is known in advance.

2 Preliminaries

We consider undirected and simple graphs, where $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges of a graph G. For a given subset S of $V(G)$, $G[S]$ denotes the subgraph of G induced by S, and $G - S$ denotes the graph $G[V(G) \setminus S]$. S is an *independent set* if $G[S]$ is an empty graph, and S is a *clique* if $G[S]$ is a complete graph. The set of neighbors of a vertex v in G is denoted by $N_G(v)$, and the set of neighbors of a vertex set S is $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$.

K_k denotes the complete graph on k vertices and C_k denotes the chordless cycle on k vertices. A *coloring* of a graph G is a function f assigning a color to each vertex of G such that adjacent vertices have different colors. A k-coloring of a graph uses at most k colors, and the smallest number of colors in a coloring of G is denoted by $\chi(G)$. The maximum size of a clique in a graph G is denoted by $\omega(G)$

Given a mapping $\varphi : V(G) \longrightarrow V(H)$ and a set $S \subseteq V(H)$, we denote by $\varphi^{-1}(S)$ the set of all those vertices of G that are mapped to a vertex of S.

The notion of treewidth was introduced by Robertson and Seymour. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\},\ T)$, where $\{X_i : i \in I\}$ is a collection of subsets of G (these subsets are called *bags*) and $T = (I, F)$ is a tree such that the following three conditions are satisfied:

1. $\bigcup_{i \in I} X_i = V(G)$.
2. For all $\{v, w\} \in E(G)$, there is an $i \in I$ such that $v, w \in X_i$.
3. For all $i, j, k \in I$, if j is on a path in T from i to k then $X_i \cap X_k \subseteq X_j$.

The *width* of a tree decomposition $(\{X_i : i \in I\},\ T)$ is $\max_{i \in I} |X_i| - 1$. The *treewidth* of a graph G, denoted by $\mathbf{tw}(G)$, is the minimum width over all its tree decompositions. A tree decomposition of width $\mathbf{tw}(G)$ is called an *optimal* tree decomposition.

3 Homomorphism to cycles

Recall that $\mathrm{HOM}(G, C_k)$ is solvable in polynomial time if k is even and NP-complete if k is odd. We study the case when $k \geq 5$ is an odd integer. Throughout the remainder of this section we assume the input graph G to be non bipartite, since every bipartite graph is homomorphic to K_2, and thus also homomorphic to C_k for all $k \geq 3$.

For a given graph G and a vertex subset $S \subseteq V(G)$, we define the levels of breadth first search starting at S as follows:

- $L_0(S) = S$;
- $L_i(S) = N_G(L_{i-1}(S)) \setminus \bigcup_{j<i} L_j(S)$, for $i > 0$.

Lemma 1. *Let $k \geq 3$ be an odd integer. A non bipartite graph $G = (V, E)$ is homomorphic to C_k if and only if there is a set $S \subset V$ such that*

- $|S| \leq |V(G)|/k$,
- *the levels $L_0(S), L_1(S), L_2(S), \ldots, L_{\lfloor \frac{k}{2} \rfloor - 1}(S)$ are independent sets in G,*
- *the graph $G - S$ is bipartite, and*
- *there is a coloring of vertices of $L_1(S), L_2(S), \ldots, L_{\lfloor \frac{k}{2} \rfloor}(S)$ in Red and Blue such that every two adjacent vertices from different levels have the same color, and every two adjacent vertices from $L_{\lfloor \frac{k}{2} \rfloor}(S)$ have different colors.*

Proof. Let us choose a vertex $v \in V(C_k)$ and let $R = (v, r_1, r_2, \ldots, r_{\lfloor k/2 \rfloor})$ and $B = (v, b_1, b_2, \ldots, b_{\lfloor k/2 \rfloor})$ be the two edge disjoint paths in C_k of length $\lfloor k/2 \rfloor$ starting at v.

Let G be homomorphic to C_k. Since G is not bipartite, every homomorphism from G to C_k is surjective. Hence there is a homomorphism τ from G to C_k such that $|\tau^{-1}(v)| \leq |V(G)|/k$. We define $S = \tau^{-1}(v)$. We then choose a homomorphism $\varphi\colon G \longrightarrow C_k$ that minimizes

$$\sum_{1 \leq i \leq \lfloor k/2 \rfloor} \frac{|\varphi^{-1}(r_i)| + |\varphi^{-1}(b_i)|}{i} \tag{1}$$

subject to $\varphi^{-1}(v) = S$.

By (1), every vertex of $\varphi^{-1}(r_i)$, $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor - 1\}$, is adjacent in G to a vertex of $\varphi^{-1}(r_{i-1})$. In fact, suppose on the contrary that there is a vertex $x \in \varphi^{-1}(r_i)$ that is not adjacent in G to any vertex of $\varphi^{-1}(r_{i-1})$. Then there is a homomorphism ϕ from G to C_k such that $\phi(y) = \varphi(y)$ for all $y \neq x$, and $\phi(x) = r_{i+2}$ if $i \leq \lfloor k/2 \rfloor - 2$ and $\phi(x) = b_{\lfloor k/2 \rfloor}$ if $i = \lfloor k/2 \rfloor - 1$. But the existence of such a homomorphism contradicts (1). By similar arguments, every vertex of $\varphi^{-1}(b_i)$ $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor - 1\}$ is adjacent to a vertex of $\varphi^{-1}(b_{i-1})$.

Thus for every $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor - 1\}$, the vertices of $\varphi^{-1}(r_i) \cup \varphi^{-1}(b_i)$ form the level $L_i(S)$ of breadth first search starting at S in G. Furthermore, each of these sets is an independent set. The graph $G - S$ is bipartite because it is homomorphic to a path. For $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor\}$, we color the vertices of $\varphi^{-1}(r_i)$ in Red and the vertices of $\varphi^{-1}(b_i)$ in Blue. Such a coloring satisfies the conditions of the lemma.

Now suppose that there is a vertex set $S \subseteq V(G)$ and a breadth first search starting at S satisfying the conditions of the lemma. We construct a homomorphism from G to C_k by mapping S to v. For $i \in \{1, 2, \ldots, \lfloor k/2 \rfloor - 1\}$, all Red vertices from level $L_i(S)$ are mapped to r_i and all Blue vertices from level $L_i(S)$ are mapped to b_i. For $i \geq \lfloor k/2 \rfloor$, Red vertices from level $L_i(S)$ are mapped to $r_{\lfloor k/2 \rfloor}$ and Blue vertices from level $L_i(S)$ are mapped to $b_{\lfloor k/2 \rfloor}$. □

We need the following algorithmic version of the result from [14] which is due to Byskov [5].

Proposition 2 ([5]). *All maximal independent sets in a triangle-free graph on n vertices can be listed in time $\mathcal{O}^*(2^{n/2})$.*

Lemma 3. *For a given graph G on n vertices, $\mathrm{HOM}(G, C_k)$ can be solved in $\mathcal{O}^*(\binom{n}{n/k})$ time.*

Proof. By Lemma 1, a non bipartite graph G is homomorphic to C_k if and only if there is a set $S \subseteq V(G)$ satisfying the conditions of the lemma.

For a given (independent) set S, one can decide whether S satisfies the conditions of Lemma 1 as follows.

1. Find the levels $L_0(S), L_1(S), L_2(S), \ldots, L_m(S)$ of breadth first search at S. If all sets $L_0(S), L_1(S), L_2(S), \ldots, L_{\lfloor \frac{k}{2} \rfloor - 1}(S)$ are independent sets in G proceed to step 2.

2. Check if $G - S$ is bipartite. If it is bipartite proceed to step 3.

3. To decide whether there is a coloring of $L_1(S) \cup L_2(S) \cup \cdots \cup L_{\lfloor \frac{k}{2} \rfloor}(S)$ which meets the condition of Lemma 1, we reduce the problem to 2-SAT as follows. We encode every vertex x of $L_1(S) \cup L_2(S) \cup \cdots \cup L_{\lfloor \frac{k}{2} \rfloor}(S)$ by a boolean variable x such that $x = \text{TRUE}$ means that vertex x is colored Red, and variable $x = \text{FALSE}$ means that vertex x is colored Blue. Every edge $\{x, y\}$ between $L_i(S)$ and $L_{i+1}(S)$, for each $1 \leq i \leq \lfloor \frac{k}{2} \rfloor - 1$, is encoded by two clauses $(\bar{x} \vee y)$ and $(x \vee \bar{y})$. This forces vertex x and vertex y to receive the same color. Every edge $\{u, v\}$ with both endpoints in $L_{\lfloor \frac{k}{2} \rfloor}(S)$ is encoded by two clauses $(u \vee v)$ and $(\bar{u} \vee \bar{v})$. This forces vertex u and vertex v to receive opposite colors. The corresponding 2-SAT formula is satisfiable if and only if S satisfies the conditions of Lemma 1 and there is a homomorphism from G to C_k that can be derived from S.

Consequently, for each given set S, constructing a homomorphism from G to C_k using S or concluding that S cannot be used can be done by solving the corresponding 2-SAT formula, and thus requires polynomial time. There are less than $(n/k)\binom{n}{n/k}$ different subsets S of size at most n/k. Hence the total running time is $\mathcal{O}^*(\binom{n}{n/k})$. □

The following algorithm improves upon the running time of the previous one for $k \in \{5, 7, 9\}$.

Lemma 4. *For a given graph G on n vertices, and an odd integer $k \geq 5$, $\text{HOM}(G, C_k)$ can be solved in $\mathcal{O}^*(2^{n/2}) = \mathcal{O}^*(1.41422^n)$ time.*

Proof. We may assume that $G = (V, E)$ is not bipartite. Furthermore C_k is 3-colorable and triangle-free for every odd integer $k \geq 5$. Thus G homomorphic to C_k implies that G is 3-colorable and triangle-free.

Let $v_1, v_2, v_3, \ldots, v_{k-1}, v_k$ be the vertices of C_k, with v_i adjacent to v_{i+1} (where indices are taken modulo k). We choose the following maximal independent set of C_k: $U = \{v_2, v_4, \ldots, v_{k-3}, v_{k-1}\}$. Suppose there is a homomorphism $\varphi : G \longrightarrow C_k$. Then $\varphi^{-1}(U)$ is an independent set of G. We claim that in this case there is even a homomorphism $\psi : G \longrightarrow C_k$ such that $\psi^{-1}(U)$ is a maximal independent set of G. Let $x \in V(G) \setminus \varphi^{-1}(U)$ such that $\{x\} \cup \varphi^{-1}(U)$ is an independent set of G, and let y be a neighbor of x in G. Then $\{x, y\} \in E(G)$ implies $\{\varphi(x), \varphi(y)\} = \{v_1, v_k\}$. Thus the following modification of φ is a homomorphism from G to C_k. Let $I' \subseteq V(G) \setminus \varphi^{-1}(U)$ such that $I = I' \cup \varphi^{-1}(U)$ is a maximal independent set of G. We define a homomorphism $\psi : G \longrightarrow C_k$ such that $\psi^{-1}(U) = I'$. For every vertex

$v \in V(G) \setminus I'$, we let $\psi(v) = \varphi(v)$. For every vertex $v \in I'$, we let $\psi(v) = v_2$ if $\varphi(v) = v_k$, and we let $\psi(v) = v_{k-1}$ if $\varphi(v) = v_1$.

The goal of our algorithm is to test, for every maximal independent set I of G, whether there is a homomorphism $\psi : G \longrightarrow C_k$ such that $\psi^{-1}(U) = I$. By the above claim, ψ must exist if G is homomorphic to C_k. For every maximal independent set I in G the test is done as follows: First, if $G - I$ is not bipartite, then reject I since a non bipartite graph cannot be homomorphic to $H - U$ which consists of a K_2 and $(k-1)/2$ isolated vertices. If $G - I$ is bipartite, let A be the set of isolated vertices of $G - I$, and let J be the set of vertices in connected components of $G - I$ that have at least two vertices. Clearly $V(G) = I \cup A \cup J$. Furthermore, since G is not bipartite, $J \neq \emptyset$.

Every vertex of J must be mapped to v_1 or v_k since each component of $G[J]$ has at least two vertices. Then every vertex of $N(J)$ must be mapped to v_2 or v_{k-1}. Clearly $N(J) \subseteq I$. Following Lemma 1, we map the vertices of G in a breadth first search manner starting from J, with levels $L_0(J) = J$, $L_1(J) = N(J)$, $L_2(J)$, ..., $L_{(k-1)/2}(J)$. At any stage we consider only the vertices that have to be mapped due to adjacencies in G to already mapped vertices. Therefore the vertices of $L_2(J)$ must be mapped to v_3 or v_{k-2}. Clearly $L_2(J) \subseteq A$. The vertices of $L_3(J)$ must be mapped to v_4 or v_{k-3}, ..., the vertices of $L_{(k-3)/2}(J)$ must be mapped to $v_{(k-1)/2}$ or $v_{(k+3)/2}$. Finally, the vertices of $L_{(k-1)/2}(J)$ must be mapped to $v_{(k+1)/2}$. Now, there may be some remaining vertices of G that are not assigned to any vertex of H by the above procedure. If $(k+1)/2$ is even, then all remaining vertices should be mapped to $v_{(k-1)/2}$ or $v_{(k+3)/2}$ if they belong to A, and to $v_{(k+1)/2}$ if they belong to I. If $(k+1)/2$ is odd, then we should do the reverse: the remaining vertices should be mapped to $v_{(k+1)/2}$ if they belong to A and to $v_{(k-1)/2}$ or $v_{(k+3)/2}$ if they belong to I. Consequently, in the end, vertices of $A \cup J$ are mapped to $V(H) \setminus U$, and vertices of I are mapped to U.

To check whether our partial mapping can be transformed into a homomorphism we shall use a 2-SAT formula. For all vertices of G except those mapped to $v_{(k+1)/2}$ there is a choice between two vertices of the host graph C_k. Furthermore adjacent vertices of G must be mapped to adjacent vertices of C_k. For every vertex x of G with $\varphi(x) \in \{v_i, v_{k-i+1}\}$ we define a boolean variable x such that variable $x =$ TRUE means that vertex x is mapped to v_i with $i = 1, 2, \ldots, (k-1)/2$, and variable $x =$ FALSE means that vertex x is mapped to v_i with $i = (k+3)/2, (k+5)/2, \ldots, k$. For each edge $\{x, y\} \in E(G[J])$ either $\varphi(x) = v_1$ and $\varphi(y) = v_k$ or vice versa. Otherwise, for each edge $\{x, y\} \in E(G)$ with $\{x, y\} \not\subseteq J$ either $\varphi(x) = v_i$ and $\varphi(y) = v_j$ with $i, j \in \{1, 2, \ldots, (k-1)/2\}$, or $\varphi(x) = v_i$ and $\varphi(y) = v_j$ with $i, j \in \{(k+3)/2, \ldots, k\}$. Therefore, for each edge $\{x, y\} \in E(G[J])$ $N_G(J)$ we insert the following two clauses in our 2-SAT formula: $(\bar{x} \vee y)$ and $(x \vee \bar{y})$. For all other edges $\{x, y\} \in E(G)$, i.e. at least one of x and y does not belong to J, we insert the following two clauses in our 2-SAT formula: $(\bar{x} \vee \bar{y})$ and $(x \vee y)$.

The corresponding 2-SAT formula is satisfiable if and only if there is a homomorphism φ from G to C_k such that $\varphi^{-1}(U) = I$. Consequently, for each maximal independent set I of G, constructing a homomorphism from G to C_k using I or

concluding that I cannot be used can be done by solving the corresponding 2-SAT formula, and thus requires linear time (see [1]). By Proposition 2, the number of maximal independent sets in a triangle free graph on n vertices is at most $2^{n/2}$ and all maximal independent sets of a triangle free graph can be enumerated in time $\mathcal{O}^*(2^{n/2})$. Thus the overall running time of our algorithm is $\mathcal{O}^*(2^{n/2})$. □

The algorithm of Lemma 3 has running time $\mathcal{O}^*(1.64939^{|V(G)|})$ when $k = 5$, $\mathcal{O}^*(1.50700^{|V(G)|})$ when $k = 7$, and $\mathcal{O}^*(1.41742^{|V(G)|})$ when $k = 9$, and its running time is $\mathcal{O}^*(\alpha^{|V(G)|})$ with $\alpha < \sqrt{2}$ for all $k \geq 11$. Hence the algorithm of Lemma 3 is faster for all $k \geq 11$, and the algorithm of Lemma 4 is faster for $k \in \{5, 7, 9\}$. Combining Lemmata 3 and 4 we obtain the following theorem.

Theorem 5. *For a given graph G on n vertices and an odd integer $k \geq 5$,* $\mathrm{HOM}(G, C_k)$ *can be solved in $\mathcal{O}^*(\min\{\binom{n}{n/k}, 2^{n/2}\})$ time.*

4 Homomorphisms to graphs of bounded treewidth

A tree decomposition $(\{X_i : i \in I\},\ T)$ of a graph G is said to be *nice* if a root of T can be chosen such that every node $i \in I$ of T has at most two children in the rooted tree T, and

1. if a node $i \in I$ has two children j_1 and j_2 then $X_i = X_{j_1} = X_{j_2}$. (i is called a **join** node.)
2. if a node $i \in I$ has one child j, then either $X_i \subset X_j$ and $|X_i| = |X_j| - 1$ (i is called a **forget** node), or $X_j \subset X_i$ and $|X_j| = |X_i| - 1$ (i is called an **introduce** node).
3. If a node $i \in I$ is a leaf of T, then $|X_i| = 1$. (i is called a **leaf** node.)

Given a nice tree decomposition $(\{X_i : i \in I\},\ T)$, we denote by T_i the subtree of T rooted at node i, for each $i \in I$. The parent of node i is denoted by $p(i)$.

It is known that every graph of treewidth at most k has a nice tree decomposition $(\{X_i : i \in I\},\ T)$ of width k such that $|I| = O(k \cdot |V|)$. Furthermore, given a tree decomposition of a graph G of width k, a nice tree decomposition of G can be computed in time $O(n)$.

There is an $\mathcal{O}^*(1.9601^n)$ algorithm to compute the treewidth and an optimal tree decomposition of a given graph [10]. There is also a well-known linear-time algorithm to compute the treewidth and an optimal tree decomposition for graphs of bounded treewidth [2].

We present an algorithm to decide whether for given graphs G and H there is an homomorphism from G to H. The algorithm is based on dynamic programming on a nice tree decomposition of H.

Theorem 6. *There is an $\mathcal{O}^*((2 \cdot \mathbf{tw}(H) + 1)^{|V(G)|})$ time algorithm taking as input a graph G, a graph H and an optimal tree decomposition of H, that solves* $\mathrm{HOM}(G, H)$ *and produces a homomorphism $\varphi : G \longrightarrow H$ if the answer is yes.*

Proof. First our algorithm transforms the given optimal tree decomposition of H into a nice tree decomposition $(\{Y_i : i \in J\},\ U)$ of width $k := \mathbf{tw}(H)$. Then we modify this nice tree decomposition as follows. For every non leaf node $i \in J$ of tree U we add a new **nochange** node i' as the parent of i, and we let the old parent of i in tree U become the parent of i' in the new tree. We let $X_{i'} = X_i = Y_i$. In this way we obtain a new nice tree decomposition $(\{X_i : i \in I\},\ T)$ of H of width $k := \mathbf{tw}(H)$. In the new tree T, the parent of every node of U is a **nochange** node, which is more convenient for our following argumentation, because there is a difference of at most one vertex between a child and the parent of a node i in T.

We define two auxiliary subsets of vertices of H for each node $i \in I$ of T: $V_i = \cup_{j \in T_i} X_j$, and $\tilde{X}_i = X_i \cap X_{p(i)}$. Notice that $\tilde{X}_i = X_i$ if $p(i)$ is an introduce, join, or nochange node, and that $\tilde{X}_i = X_i \backslash \{u\}$ if $p(i)$ is a forget node with $X_{p(i)} = X_i \backslash \{u\}$. For r, the root of T, we define $\tilde{X}_r = X_r$.

Our algorithm computes for each node $i \in I$ of T in a bottom-up fashion all characteristics of i, defined as follows.

Definition. *A tuple $(S; (v_1, S_1), (v_2, S_2), \ldots, (v_{l_i}, S_{l_i}); i)$ is a characteristic of node $i \in I$ of T if $S \subseteq V(G)$ and $\{S_1, S_2, \ldots, S_{l_i}\}$ is a partition of S such that there is a homomorphism $\varphi : G[S] \longrightarrow H[V_i]$ that satisfies the following two conditions.*

- $\tilde{X}_i = \{v_1, v_2, \ldots, v_{l_i}\}$;
- *For every $j \in \{1, 2, \ldots, l_i\}$, $\varphi^{-1}(v_j) = S_j$.*

Notice that characteristics are defined in such a way that G is homomorphic to H if and only if there is at least one characteristic for the root r of T. Furthermore the number of characteristics of a node of T is at most $\sum_{i=0}^{|V(G)|} \binom{n}{i} \cdot k^i = (1+k)^{|V(G)|}$.

For each forget, introduce, nochange, and join node $i \in I$ of T, our algorithm computes by dynamic programming all characteristics $(S; (v_1, S_1), \ldots, (v_{l_i}, S_{l_i}); i)$ of i using the full set of characteristics of i's children. Thus it suffices to describe how the full set of characteristics can be computed from the characteristics of the children for the different types of nodes in T.

Leaf node:
Let i be a leaf node, thus $X_i = \{u\}$ for some vertex u of H. For a subset S of $V(G)$, there is a homomorphism φ from $G[S]$ to $H[V_i]$ with $V_i = \{u\}$ if and only if $\varphi^{-1}(\{u\}) = S$, and hence S is an independent set. Thus $(S; (u, S); i)$ is a characteristic of the leaf node i if and only if S is an independent set of G.

Introduce node:
Let i be an introduce node with child j. Thus $X_i = X_j \cup \{u\}$ for some vertex $u \in V(H) \setminus V_j$, and consequently $\tilde{X}_j = X_j$. Notice that the parent of i is a nochange node, and thus $X_i = X_{p(i)}$ and $\tilde{X}_i = \tilde{X}_j \cup \{u\}$.

All characteristics of node i can be obtained by extending a characteristic of j. Since $\tilde{X}_i = \tilde{X}_j \cup \{u\}$, each characteristic of i obtained from $(S; (v_1, S_1), \ldots, (v_{l_j}, S_{l_j}); j)$ is of the form $(S \cup S'; (v_1, S_1), \ldots, (v_{l_j}, S_{l_j}), (u, S'); i)$ where $S' \subseteq V(G) \setminus S$ is an independent set in G, and for all $x \in N_{G[S]}(S')$, $\varphi(x) \in N_H(u)$. These conditions

can be checked in polynomial time. Finally one characteristic of j extends to at most $2^{|V(G)|-|S|}$ characteristics of i, since S' must be an independent set of $G - S$. Therefore we compute at most $\sum_{i=0}^{n} \binom{n}{i} \cdot k^i \cdot 2^{n-i} = (1+2k)^n$ characteristics to obtain a full set of characteristics of an introduce node.

Forget node:
Let i be a forget node with child j. Thus $X_i = X_j \setminus \{u\}$ for some vertex $u \in X_j$, and consequently $\tilde{X}_j = X_i$. The parent of i is a nochange node, and thus $X_i = X_{p(i)}$ and $\tilde{X}_i = \tilde{X}_j$.

$\tilde{X}_i = \tilde{X}_j$ implies that each characteristic of i can be obtained directly from a characteristic of j by simply replacing j with i. Thus each characteristic of j is a characteristic of i.

Nochange node:
Let i be nochange node with child j. Thus $X_i = X_j$. If the parent of i is a forget node then $X_{p(i)} = X_i \setminus \{u\}$ for some vertex $u \in X_i$, and thus $\tilde{X}_i = \tilde{X}_j \setminus \{u\}$. If the parent of i is an introduce or join node, then $\tilde{X}_i = \tilde{X}_j$.

If $\tilde{X}_i = \tilde{X}_j$ then each characteristic of i extends into one characteristic of j by simply replacing j by i. Otherwise, $\tilde{X}_i = \tilde{X}_j \setminus \{u\}$ implies that each characteristic of i can be obtained from a characteristic of j, say $(S; (v_1, S_1), \ldots, (v_{l_j}, S_{l_j}); j)$, by simply removing the pair (v_t, S_t) where $u = v_t$. One obtains $(S; (v_1, S_1), \ldots, (v_{t-1}, S_{t-1}), (v_{t+1}, S_{t+1}), \ldots, (v_{l_i}, S_{l_i}); i)$.

Thus again each characteristic of j extends into one characteristic of i.

Join node:
This is the most interesting node type. Let i be a join node with children j_1 and j_2; thus $X_i = X_{j_1} = X_{j_2}$. The parent of i is a nochange node, thus $\tilde{X}_i = \tilde{X}_{j_1} = \tilde{X}_{j_2} = X_i$.

Let $(S'; (v_1, S_1), \ldots, (v_{l_{j_1}}, S_{l_{j_1}}); j_1)$ be a characteristic of j_1. It extends into a characteristic of node i if there is a characteristic $(S''; (v_1, S_1), \ldots, (v_{l_{j_2}}, S_{l_{j_2}}); j_2)$ of j_2. I.e., both characteristics have the same set of pairs (v_i, S_i) which requires that $l_i = l_{j_1} = l_{j_2}$. In this case $(S' \cup S''; (v_1, S_1), \ldots, (v_{l_i}, S_{l_i}); i)$ is a characteristic of i, if there are no edges between $S' \setminus S''$ and $S'' \setminus S'$ in G.

Thus we compute characteristics $(S' \cup S''; (v_1, S_1), \ldots, (v_{l_i}, S_{l_i}); i)$ of i, for each subset $S' \cup S''$ of $V(G)$, each partition of S into at most k subsets, and any choice of a subset S'. Therefore we compute at most $\sum_{i=0}^{n} \binom{n}{i} \cdot k^i \cdot 2^i = (1+2k)^n$ characteristics to obtain a full set of characteristics of a join node.

Finally, notice that the number of nodes in the decomposition is a polynomial in $|V(H)|$, and that suitable data structures guarantee that the characteristics of a node can be stored such that find and insert operations can be done in polynomial time. Thus the overall running time of our algorithm is $\mathcal{O}^*((2 \cdot \mathbf{tw}(H)+1)^{|V(G)|})$. □

5 Concluding remarks and open questions

For given graphs G and H, in what time can we solve HOM(G, H)? The trivial solution brings us $\mathcal{O}(|V(G)|^{|V(H)|})$ running time. There is a randomized $\mathcal{O}((0.4518 \cdot |V(H)|)^{|V(G)|})$ time algorithm solving HOM(G, H) which is a consequence of a more general result on constraint satisfaction problems [4].

In this paper we observed that if the right side graph H is of bounded treewidth, then HOM(G, H) can be solved in time $c^{|V(G)|} \cdot |V(H)|^{O(1)}$ for some constant c. It is natural to ask then if for any graphs G and H the problem HOM(G, H) is solvable with running times (1) $f(|V(H)|) \cdot |V(G)|^{O(1)}$, or (2) $f(|V(G)|) \cdot |V(H)|^{O(1)}$ for some computable function $f\colon N \to N$. (Unfortunately) the answer to each of the questions is negative up to some widely believed assumptions in complexity theory.

In fact for case (1): an $f(|V(H)|) \cdot |V(G)|^{O(1)}$ time algorithm is also a polynomial time algorithm for the NP-complete 3-coloring problem implying $P = NP$.

To answer the second question, we use the widely believed assumption from parameterized complexity [9] that p-clique is not fixed parameter tractable, or in other words that there is no algorithm finding a clique of size p in a graph on h vertices in time $f(p) \cdot h^{O(1)}$ unless $FPT = W[1]$, a collapse of a parameterized hierarchy considered to be very unlikely. Since K_p is homomorphic to H if only if H has a clique of size at least p, the HOM(K_p, H) problem is equivalent to finding a p-clique in H. Therefore, the existence of an $f(|V(G)|) \cdot |V(H)|^{O(1)}$ time algorithm for HOM(G, H) would imply that p-clique is fixed parameter tractable, thus $FPT = W[1]$.

Now our question is whether a running time $\mathcal{O}((c \cdot |V(H)|)^{|V(G)|})$ for some constant c is the best we can hope for to solve HOM(G, H)? Can it be solved, say by an $\mathcal{O}(c^{|V(G)|+|V(H)|} \cdot |V(G)|^{O(1)} \cdot |V(H)|^{O(1)})$ time algorithm?

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