On the Structure of (P_5,Gem)-Free Graphs *

Andreas Brandstädt[†] Dieter Kratsch[‡]

October 21, 2002

Abstract

We give a complete structure description of (P_5,gem)-free graphs. By the results of a related paper, this implies bounded clique width for this graph class. Hereby, as usual, the P_5 is the induced path with five vertices a, b, c, d, e and four edges ab, bc, cd, de, and the *gem* consists of a P_4 a, b, c, d with edges ab, bc, cd plus a universal vertex e adjacent to a, b, c, d.

Keywords: (P_5,gem)-free graphs; modules and homogeneous sets in graphs; prime graphs; modular decomposition; clique width.

1 Introduction

Recently, in connection with graph grammars, Courcelle, Engelfriet and Rozenberg in [6] introduced the notion of clique width of a graph which is closely related to modular decomposition. The concept of clique width of a graph has attracted much attention due to the fact that, in [7], Courcelle, Makowsky and Rotics have shown that every graph problem definable in LinEMSOL($\tau_{1,L}$) (a variant of Monadic Second Order Logic) is linear-time solvable on graphs with bounded clique width if a k-expression describing the input graph is given. The problems Vertex Cover, Maximum Weight Stable Set (MWS), Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL($\tau_{1,L}$) definable problems.
It is known that the class of P_4-free graphs is exactly the class of graphs having clique width at most 2, and a 2-expression can be found in linear time along the cotree of a cograph. Due to the basic importance of cographs, it is of interest to consider classes defined by forbidden one-vertex extensions of a P_4 - see Figure 1.
We follow this line of research by investigating the structure of (P_5,gem)-free graphs; in [1], it is shown that these structure results imply that (P_5,gem)-free graphs (and thus also their complements, the (house,co-gem)-free graphs) have bounded clique width.

Another motivation for studying the structure of (P_5,gem)-free graphs is the fact that for P_5-free graphs, the complexity of the Maximum Stable Set Problem is unknown, and a recent result of Mosca [14] shows that the Maximum Stable Set Problem can be solved in polynomial time for (P_5,gem)-free graphs. Our structure results imply that the MWS problem can be solved on (P_5,gem)-free graphs in a more efficient way, and moreover, bounded clique width of this class implies polynomial time bounds for many other problems

*Research of the first author supported by LORIA Nancy, France

†Fachbereich Informatik, Universität Rostock, A.-Einstein-Str. 21, D-18051 Rostock, Germany, ab@informatik.uni-rostock.de

‡Université de Metz, Laboratoire d'Informatique Théorique et Appliquée, 57045 Metz Cedex 01, France, kratsch@lita.sciences.univ-metz.fr

on this class.

This paper is organized as follows:

- In section 2, we give some basic graph notions and mention the concept of clique width and some basic related results.
- In section 3, we describe the structure of prime (P_5,gem)-free graphs containing $2K_2$ (prime (house,co-gem)-free graphs containing C_4).
- In section 4, we describe the structure of prime (C_4,co-gem)-free graphs containing C_6.
- In section 5, we describe the structure of prime (C_4,C_6,co-gem)-free graphs containing C_5.
- In section 6, we summarize the obtained results in a structure theorem characterizing the (P_5,gem)-free graphs.

2 Basic notions and preliminary results

Let $G = (V, E)$ be a finite undirected graph, and let $|V| = n$ and $|E| = m$. Let $N(v) := \{u : u \in V, u \neq v, uv \in E\}$ denote the *open neighborhood* of v and $N[v] := N(v) \cup \{v\}$ the *closed neighborhood* of v. For $U \subseteq V$ let $G(U)$ denote the subgraph of G induced by U. Throughout this paper, all subgraph containments are understood as induced subgraph containments. If for $U \subset V$, a vertex not in U is adjacent to exactly k vertices in U then it is called *k-vertex* for U.
A vertex set $U \subseteq V$ is a *clique* in G if the vertices in U are pairwise adjacent. The *complement graph* $\overline{G} = (V, \overline{E})$ of G is defined by $\overline{E} = \{uv : u, v \in V, u \neq v \text{ and } uv \notin E\}$. $\overline{G}$ is also denoted by co-G. A vertex set $U \subseteq V$ is *stable* (*independent*) in G if U is a clique in $\overline{G}$.
A vertex set $V' \subseteq V$ is a *dominating set* in G, if for all vertices $u \in V \setminus V'$, there is a vertex $v \in V'$ such that $uv \in E$.
For $k \geq 1$, let P_k denote a chordless path with k vertices and $k - 1$ edges, and for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *hole* is a C_k for $k \geq 5$.
Figure 1 describes all possible P_4 extensions by one vertex, such as P_5, chair, P, C_5, bull, gem and their complements. Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$.
For disjoint vertex sets X and Y, the *join* (*co-join*) operation between X and Y adds all edges (non-edges) between all vertex pairs $x \in X$ and $y \in Y$.
A vertex set $M \subseteq V$ is a *module* in G if for all vertices $x \in V \setminus M$, x has either a join or a co-join to M. The *trivial modules* of G are $\emptyset, V$ and the singletons. A *homogeneous set* in G is a nontrivial module in G. A graph containing no homogeneous set is called *prime*. Note that the smallest prime graph with at least three vertices is the P_4.
A homogeneous set M is *maximal* if no other homogeneous set properly contains M. It is well-known that in a connected graph $G = (V, E)$ with connected complement $\overline{G}$, the maximal homogeneous sets are pairwise disjoint (see e.g. [13]). This fact leads to the uniquely determined *modular decomposition tree* since every vertex of G is contained in a uniquely determined maximal homogeneous set if there is any such set containing the vertex. The leaves of the tree are the vertices of G, and the internal nodes of the tree are either join or co-join operations or prime nodes representing a prime subgraph (see e.g. [13]) obtained in the following way: The graph G^* obtained from graph G by

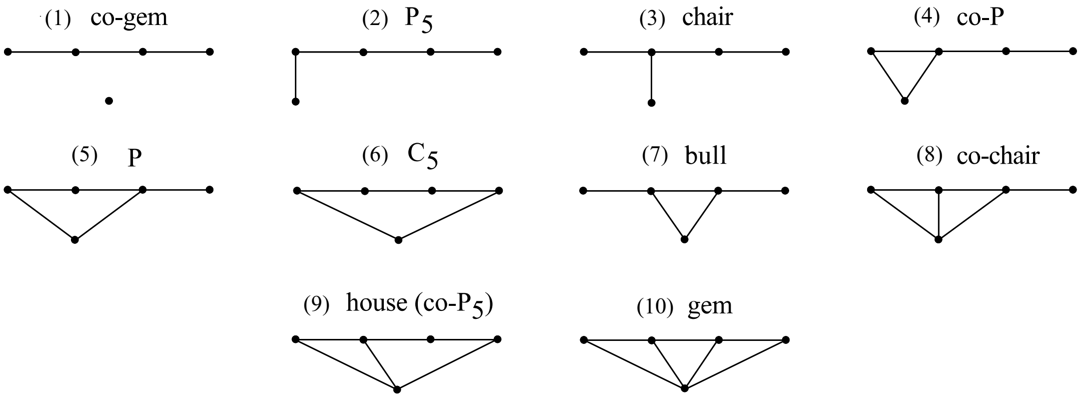

Figure 1: All one-vertex extensions of a P_4

contracting every maximal homogeneous set of G to a single vertex is called the *characteristic graph* of G. It is well-known that if G is connected and co-connected then G^* is prime.

The modular decomposition tree is of basic importance for many algorithmic applications, and in [13, 8, 9], linear time algorithms are given for determining the modular decomposition tree of an input graph.
A graph is a *cograph* if in its modular decomposition tree, all internal nodes are join or co-join nodes i.e. the graph can be recursively constructed from single vertices by a sequence of join and co-join operations.
Cographs are exactly the P_4-free graphs, and their modular decomposition trees were called *cotrees*. The cotree representation allows to solve various NP-hard problems in linear time when restricted to cographs, among them the problems Maximum (Weight) Stable Set and Maximum (Weight) Clique.
See [3, 4, 5] for more information on cographs and [2] for a survey on this graph class and others as well as [5] for a linear time recognition of P_4-free graphs (which is part of the linear time algorithms in [13, 8]).

Subsequently we will make use of the following

Lemma 2.1 ([11]) *If a prime graph contains a C_4 then it contains a $\overline{P_5}$ (i.e. house) or A or domino.*

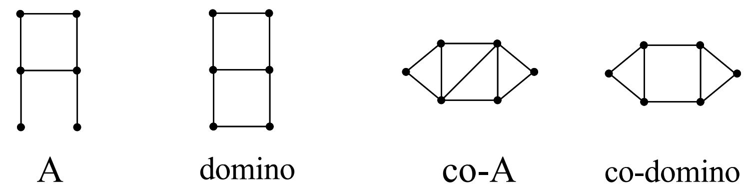

Figure 2: The graphs A and domino with their complements

In its graph complement version, Lemma 2.1 means the following:

Corollary 2.1 *If a prime graph contains a $2K_2$ then it contains a P_5 or $\overline{A}$ or co-domino.*

Note that the graph $\overline{A}$ (i.e. double-gem) contains a gem (see Figure 2). Thus it follows that the characteristic graphs of (P_5,gem)-free graphs either contain a co-domino (in which case the structure turns out to be very simple - see Lemma 3.1) or are $2K_2$-free.
Moreover, in a connected gem-free graph, homogeneous sets are P_4-free. The same holds for a co-connected co-gem-free graph.
Throughout this paper, we occasionally switch between the (P_5,gem)-free graphs and their complements being the (house,co-gem)-free graphs. Note that homogeneous sets in G are homogeneous in the complement graph $\overline{G}$ as well and the complement of a prime graph is prime.

Subsequently, we need the following classes of (prime) graphs:

- G is *matched co-bipartite* if its vertex set is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching.
- G is *co-matched bipartite* if G is the complement graph of a matched co-bipartite graph.
- A bipartite graph $B = (X, Y, E)$ is a *chain graph* [17] if for all vertices from X (Y), their neighborhoods in Y (X) are linearly ordered. If moreover, $|X| = |Y|$ and for all vertices from X (Y), their neighborhoods in Y (X) have size $1, 2, \ldots, |Y|$ ($1, 2, \ldots, |X|$) then these graphs are prime.
- G is a *co-bipartite chain graph* if it is the complement of a bipartite chain graph.

3 Prime (P_5,gem)-free graphs containing $2K_2$

Lemma 3.1 *If a prime graph G contains a $2K_2$ then G is (P_5,gem)-free if and only if G is a matched co-bipartite graph.*

Proof. It is easy to see that matched co-bipartite graphs are (P_5,gem)-free. Now assume that G is prime (P_5,gem)-free containing a $2K_2$. Then, according to Corollary 2.1, G contains a co-domino D (subsequently, we refer to its vertices $1, 2, \ldots, 6$ as in Figure 3); we call 1 and 6 the *end-vertices* of D and 2, 3, 4, 5 the *inner vertices* of D. We start with some adjacency properties with respect to D.

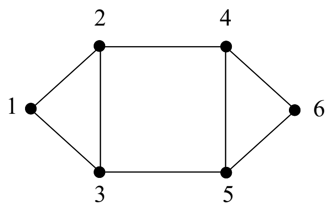

Figure 3: A co-domino

A straightforward case analysis corresponding to Figure 4 (where boldface edges indicate P_5 or gem) shows that there are only five adjacency types of a vertex to a co-domino in a (P_5,gem)-free graph:

Claim 3.1 *Let G be a (P_5,gem)-free graph and let D be a co-domino in G with vertices $1, 2, \ldots, 6$ as in Figure 3. For vertices $x \notin \{1, 2, \ldots, 6\}$ adjacent to D, there are only the following possibilities:*

(A_1) *x is a 2-vertex for D having the same neighbors as an end-vertex of D and is nonadjacent to this end-vertex;*

(A_2) *x is a 3-vertex for D having the same neighbors as an end-vertex of D and is adjacent to this end-vertex;*

(A_3) *x is a 3-vertex having the same neighbors as an inner vertex in D and is nonadjacent to this inner vertex;*

(A_4) *x is a 4-vertex having the same neighbors as an inner vertex in D and is adjacent to this inner vertex;*

(A_5) *x is a 4-vertex having the same inner neighbors as one of the end-vertices and is adjacent to both end-vertices.*

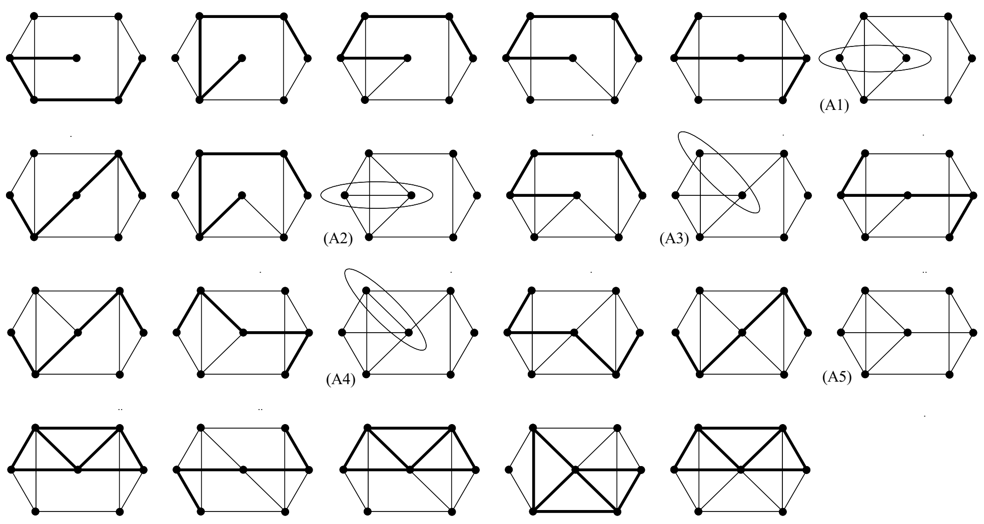

Figure 4: Adjacency of one vertex to a co-domino in a (P_5,gem)-free graph

Claim 3.2 *In a connected (P_5,gem)-free graph $G = (V, E)$, every co-domino dominates G.*

Proof. Let D be a co-domino in G. Assume that there is a vertex $y \in V$ in distance 2 to D. Then a straightforward case analysis (see Figure 5) according to the adjacency cases given in Claim 3.1 shows that in every case the vertex y together with a common neighbor x of y and D and three suitable vertices in D induce a P_5 (indicated by the boldface edges in Figure 5) which is impossible. Thus, no vertex in G has distance 2 to D. □

Next we will show that in prime graphs, the adjacency cases (A_3) and (A_4) from Claim 3.1 (see Figure 4) are impossible. We consider all vertices having the same adjacency relation as the inner vertex 2 of the co-domino D with the other vertices $1, 3, \ldots, 6$ (labeling as in Figure 3). By symmetry, the arguments are the same for the other inner vertices 3, 4, 5 of D.

Claim 3.3 *Let G be a prime (P_5,gem)-free graph and let D denote a co-domino in G, D having vertices $1, 2, \ldots, 6$ as in Figure 3. Then the set $M_2 := N(1) \cap N(3) \cap N(4) \cap \overline{N}(5) \cap \overline{N}(6)$ is a module and thus has size 1.*

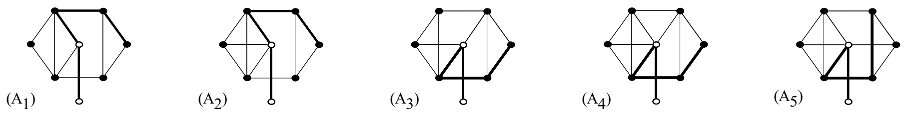

Figure 5: Co-dominoes are dominating

Proof. Note that $2 \in M_2$ and every $x \in M_2$ together with $1,3,4,5,6$ induces a co-domino D_x in G. Assume that M_2 is not a module i.e. there are $x, y \in M_2$ and $z \notin M_2 \cup \{1,3,4,5,6\}$ such that z is adjacent to x and nonadjacent to y. Note that z is a k-vertex for D_y if and only if z is a $(k+1)$-vertex for D_x. We first show that z is not a 2-vertex for D_y: If z is a 2-vertex for D_y then $z4 \in E$ and $z5 \in E$ but then $xz564$ is a gem in G - contradiction. Thus, z must be a 3-vertex for D_y and a 4-vertex for D_x.
If z is of type (A_2) for D_y then $z4 \in E$, $z5 \in E$ and $z6 \in E$ and $z1 \notin E$, and since $y1xz6$ is no P_5, $xy \in E$ but now $yxz54$ is a gem - contradiction.
If z is of type (A_3) for D_y then, since $z \notin M_2$, z has other D-neighbors than y i.e. $z3 \in E$, $z4 \in E$ and $z6 \in E$ but then $3x46z$ is a gem - contradiction. Thus, M_2 is a module and is of size one in the prime graph G. □

By symmetry, Claim 3.3 holds as well for the sets

$$M_3 := N(1) \cap N(2) \cap N(5) \cap \overline{N}(4) \cap \overline{N}(6),$$

$$M_4 := N(6) \cap N(5) \cap N(2) \cap \overline{N}(3) \cap \overline{N}(1), \text{ and}$$

$$M_5 := N(6) \cap N(4) \cap N(3) \cap \overline{N}(2) \cap \overline{N}(1).$$

Let D again be a co-domino with vertices $1, 2, \ldots, 6$ as in Figure 3. By Claim 3.2, every vertex in $G \setminus D$ is adjacent to D, and by Claim 3.3, the adjacency types (A_3) and (A_4) are impossible for vertices outside D. Thus every vertex outside D and adjacent to D is adjacent to 2 and 3 and is nonadjacent to 4 and 5 or vice versa. Let

$$L := \{v : v \in V \text{ and } v \text{ is adjacent to 2 and 3 and nonadjacent to 4 and 5}\} \text{ and}$$

$$R := \{v : v \in V \text{ and } v \text{ is nonadjacent to 2 and 3 and adjacent to 4 and 5}\}.$$

Note that $1 \in L$ and $6 \in R$. Thus $V = L \cup R \cup \{2,3,4,5\}$ is a partition of the vertex set.

Claim 3.4 *L and R are cliques.*

Proof. Assume to the contrary that L contains a pair x, x' of nonadjacent vertices. We first show that in this case x and x' have the same neighborhoods in R. Assume that there is a vertex $y \in R$ with $xy \in E$, $x'y \notin E$. Then $x'3xy4$ is a P_5 - contradiction.
This means that nonadjacent vertices in L have the same neighborhoods in R, but then the co-connected component in L containing x and x' is a homogeneous set contradicting the assumption that G is prime. Thus L and analogously R are cliques. □

Claim 3.5 *The edges between L and R form a matching.*

Proof. Assume that $x, x' \in L$ have a common neighbor $y \in R$. Then, since $\{x, x'\}$ is no homogeneous set, let x have another neighbor y' in R for which $x'y' \notin E$. Then $3x'yy'x$ is a gem - contradiction. Thus the edges between L and R form a matching. □

Since G is prime, at most one vertex in each of the sets L and R is unmatched. Altogether, this shows Lemma 3.1. □

In its complement version, Lemma 3.1 means the following:

Corollary 3.1 *Assume that the prime graph G contains a C_4. Then G is (house,co-gem)-free if and only if G is a co-matched bipartite graph.*

From now on, we consider prime ($2K_2$,gem)-free graphs and, switching to the complement, prime (C_4,co-gem)-free graphs. Note that co-gem-free graphs contain no cycles C_k, $k \geq 7$.

4 Prime (C_4,co-gem)-free graphs containing C_6

We call a graph *specific* if it is one of the three graphs in Figure 6 or one of its prime induced subgraphs.

Lemma 4.1 *If G is a prime (C_4,co-gem)-free graph containing a C_6 then G is a specific graph.*

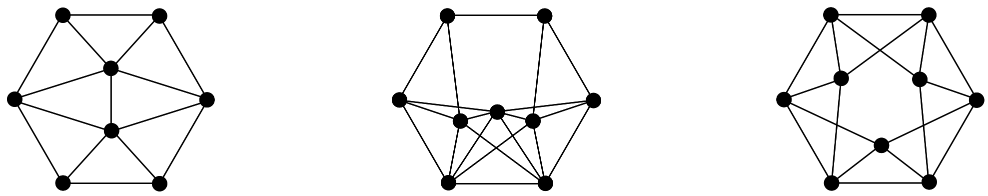

Figure 6: Prime (C_4,co-gem)-free graphs containing a C_6

Proof. Let G contain a C_6 C with vertices $1, 2, \ldots, 6$ and edges $\{i, i+1\}$, $i \in \{1, 2, \ldots, 6\}$. Subsequently in this section, all index arithmetic is done modulo 6. Since G is co-gem-free, C dominates G. A straightforward case analysis shows that there are only three admissible cases for a vertex being adjacent to C in a (C_4,co-gem)-free graph:

- 3-vertices with consecutive neighbors in C,
- 4-vertices with consecutive neighbors in C, and
- 6-vertices.

Claim 4.1 *C has no 6-vertices.*

Proof. Every 6-vertex for C would be adjacent to all 3- and 4-vertices as well since G is C_4-free. Thus, C has no 6-vertices since G is prime. □

Claim 4.2 *C has no 3-vertices.*

Proof. Assume that C has a 3-vertex v with neighbors 6, 1 and 2. Let $M_1 := N(2) \cap N(6) \cap \overline{N}(3) \cap \overline{N}(4) \cap \overline{N}(5)$. Note that $1 \in M_1$. We claim that M_1 is a module which means that $|M_1| = 1$. Assume not, i.e. there are $x, y \in M_1$, $xy \in E$, and $z \notin M_1 \cup \{2, 3, 4, 5, 6\}$ such that $xz \in E$ and $yz \notin E$. Note that x and y together with $\{2, 3, 4, 5, 6\}$ form a C_6 C_x, C_y respectively. Note that z is a $(k+1)$-vertex for C_x if and only if z is a k-vertex for C_y. Thus, the only possibility is that z is a 3-vertex for C_y and a 4-vertex for C_x. If z is adjacent to 3 then also to 2 and if z is adjacent to 5 then also to 6 since G is C_4-free. Moreover, z cannot be adjacent to both of 2 and 6 since otherwise $2y6z$ is a C_4. Assume without loss of generality that the neighbors of z are $6, 5$ and 4. Then $yxz5$ is a P_4 missing 3 i.e. $yxz53$ is a co-gem - contradiction. □

Note that if G is C_5-free then 4-vertices are impossible as well, and in this case, if G is containing a C_6 C then $G = C$ and thus, G is specific.

Claims 4.1 and 4.2 imply that every vertex outside C is adjacent to exactly four consecutive vertices of C. Let $N_{i,i+3}$ denote the set of 4-vertices adjacent to $i, i{+}1, i{+}2, i{+}3$, $i \in \{1, \dots, 6\}$. Obviously, for all $i \in \{1, \dots, 6\}$, $N_{i,i+3}$ is a clique.

Claim 4.3 *For all $i \in \{1, \dots, 6\}$, $N_{i,i+3}$ is a module and thus, $|N_{i,i+3}| \leq 1$.*

Proof. Assume without loss of generality that $N_{1,4}$ is no module. Let $x, y \in N_{1,4}$ and $z \notin N_{1,4}$ with $zx \in E$ and $zy \notin E$. Since the P_4 $y165$ must dominate z, $z1 \in E$ or $z6 \in E$ or $z5 \in E$. Since the P_4 $y456$ must dominate z, $z4 \in E$ or $z5 \in E$ or $z6 \in E$. Since $1y4z$ is no C_4, $z1 \in E$ implies $z4 \notin E$ and vice versa. Assume without loss of generality $z4 \notin E$. Since $zx45$ is no C_4, now $z5 \notin E$. Thus $z6 \in E$.
Now, since z is a 4-vertex for C, $z1 \in E$, $z2 \in E$, $z3 \in E$ follows but now $1y3z$ is a C_4 - contradiction. Thus, for all $i \in \{1, \dots, 6\}$, $N_{i,i+3}$ is a module. □

Claim 4.3 means that G has at most 12 vertices. In the next step, we show that G is specific i.e., in particular, it has at most 9 vertices. For $i \in \{1, \dots, 6\}$, let $N_{i,i+3} = \{x_i\}$ if $N_{i,i+3} \neq \emptyset$.

Claim 4.4 *For all $i \in \{1, \dots, 6\}$, $x_i x_{i+1} \in E$, $x_i x_{i+3} \in E$, and $x_i x_{i+2} \notin E$ holds.*

Proof. We show Claim 4.4 for $i = 1$. Since $2x_1 4x_2$ is no C_4, $x_1 x_2 \in E$ holds. Since $1x_1 4x_4$ is no C_4, $x_1 x_4 \in E$ holds. Since $x_1 x_3 61$ is no C_4, $x_1 x_3 \notin E$ holds. □

Assume now that x_1, x_2 exist and also x_4 exists. Then $x_1 x_2 5 x_4$ is a C_4 - contradiction. The same contradiction is obtained if x_1, x_2 and x_5 exist. Thus, if $N_{1,4} \neq \emptyset$ and $N_{2,5} \neq \emptyset$ then $N_{4,1} = \emptyset$ and $N_{5,2} = \emptyset$. This shows that, up to symmetry, there are three possibilities for the existence of a maximal subset of $\{x_1, \dots, x_6\}$ described in Figure 6:

1. x_1 and x_4 exist;
2. x_1, x_2, x_3 exist;
3. x_1, x_3, x_5 exist.

All these possibilities are specific graphs described in Figure 6. □

Note that there are specific graphs without C_6. However all specific graphs are (house,co-gem)-free.

From now on, G is a prime (C_4,C_6,co-gem)-free graph, i.e. the only possible chordless cycle in G is C_5.

5 Prime (C_4,C_6,co-gem)-free graphs containing C_5

Lemma 5.1 *Let $G = (V, E)$ be a prime (C_4,C_6,co-gem)-free graph containing a C_5. Then one of the following conditions holds:*

- (i) *G is a specific graph;*
- (ii) For every C_5 $C = (1, 2, 3, 4, 5)$ *in G, V has a partition into the vertex set $\{1, \ldots, 5\}$ of C, a clique U of 5-vertices for C and a cograph of 2-vertices for C being adjacent to the same vertices $i, i + 1 \in \{1, \ldots, 5\}$ of C.*

Corollary 5.1 *Let $G = (V, E)$ be a prime ($2K_2$,$\overline{C_6}$,gem)-free graph containing a C_5. Then one of the following conditions holds:*

- (i) *$\overline{G}$ is a specific graph;*
- (ii) For every C_5 $C = (1, 2, 3, 4, 5)$ *in G, V has a partition into the vertex set $\{1, \ldots, 5\}$ of C, a stable set A of 0-vertices for C and a cograph B of 3-vertices for C being adjacent to the same vertices $i, i + 2, i + 3 \in \{1, \ldots, 5\}$ of C.*

Proof of Lemma 5.1. Let C be a C_5 in G with vertex set $\{1, \ldots, 5\}$ and edges $\{i, i + 1\}$, $i \in \{1, \ldots, 5\}$. Since G is co-gem-free, C dominates G. Thus, C has no 0-vertices. A straightforward case analysis shows that C has also no 1- and no 4-vertices, and the only 2- and 3-vertices have consecutive neighbors in C.
Subsequently, all index arithmetic with respect to C is done modulo 5. For $i \in \{1, \ldots, 5\}$, let $N_{i,i+1}$ denote the set of 2-vertices adjacent to $i, i + 1$, let $Q_i = \{x : x = i$ or x is a 3-vertex adjacent to $i - 1, i, i + 1\}$, and let U denote the set of 5-vertices.
Obviously, for all $i \in \{1, \ldots, 5\}$, Q_i has a join to U since vertices from these sets are adjacent to the nonadjacent vertices $i - 1$ and $i + 1$ and G is C_4-free, and for the same reason, U and Q_i are cliques and there are no edges between Q_i and Q_j for $|i - j| > 1$, $i, j \in \{1, \ldots, 5\}$ since G is C_4-free. Moreover, $N_{i,i+1} \cup \{i, i + 1\}$ are (C_4-free) cographs since otherwise there would be a P_4 missing $i + 3$. We first show that at most one of the sets $N_{i,i+1}$, $i \in \{1, \ldots, 5\}$, can be nonempty.

Claim 5.1 *If for some $i \in \{1, \ldots, 5\}$, $N_{i,i+1} \neq \emptyset$ then for all $j \neq i$, $j \in \{1, \ldots, 5\}$, $N_{j,j+1} = \emptyset$ holds.*

Proof. Assume that $N_{1,2} \neq \emptyset$ and let a be a 2-vertex adjacent to 1 and 2. If there is a 2-vertex b with neighbors 2 and 3 then $ab \in E$, otherwise $a154$ is a P_4 missing b but now $ab3451$ is a C_6 - contradiction. A similar argument holds for 2-vertices adjacent to 5,1. Now assume that b is a 2-vertex adjacent to 4 and 5. If $ab \in E$ then $ab51$ is a C_4, and if $ab \notin E$ then $a15b$ is a P_4 missing 3 - contradiction. A similar argument holds for 2-vertices adjacent to 3 and 4. □

Claim 5.2 *For all $i \in \{1, \ldots, 5\}$, at least one of the edge sets between Q_i, Q_{i+1} and Q_{i+1}, Q_{i+2} is a join.*

Proof. Assume without loss of generality that there are nonedges $x_2x_3 \notin E$ and $x_3'x_4 \notin E$ for $x_i \in Q_i$, $i \in \{2,3,4\}$ and $x_3' \in Q_3$. If $x_3 = x_3'$ then $x_2 15 x_4$ is a P_4 missing x_3 - contradiction. Assume now that x_2 and x_4 have no common nonneighbor in Q_3. Then $x_3 \neq x_3'$ and $1x_2x_3'x_3x_45$ is a C_6 - contradiction. □

Claim 5.3 *At most one of the edge sets between Q_i, Q_{i+1}, $i \in \{1,\ldots,5\}$, is no join.*

Proof. Assume without loss of generality that there are nonedges $x_1x_2 \notin E$ and $x_4x_5 \notin E$ for $x_i \in Q_i$, $i \in \{1,2,4,5\}$. Then $x_2 2 x_1 x_5$ is a P_4 missing x_4 - contradiction. □

Claim 5.4 *Let $i \in \{1,\ldots,5\}$. If the edge set between Q_i and Q_{i+1} is no join then C has no 2- and no 5-vertices, and $|Q_j| = 1$ for $j \notin \{i, i+1\}$.*

Proof. We first show that C has no 2-vertices. Assume without loss of generality that there is a nonedge $x_1x_2 \notin E$ for $x_i \in Q_i$, $i \in \{1,2\}$. This defines C_5's C_{x_1} with vertices $x_1, 2, 3, 4, 5$ and C_{x_2} with vertices $1, x_2, 3, 4, 5$. First assume that a is a 2-vertex for C adjacent to 1 and 2. Then, since a is not a 1-vertex for C_{x_1}, $ax_1 \in E$, and since a is not a 1-vertex for C_{x_2}, $ax_2 \in E$ but now x_1ax_2345 is a C_6 - contradiction. Thus $N_{1,2} = \emptyset$.
Assume now that a is a 2-vertex for C adjacent to 5 and 1. Then, since a is not a 1-vertex for C_{x_1}, $ax_1 \in E$, and since $ax_1 2 x_2$ is no P_4 missing 4, $ax_2 \in E$ but now $ax_1 2 x_2$ is a C_4 - contradiction. Thus $N_{5,1} = \emptyset$. With a similar argument one shows that $N_{2,3} = \emptyset$.
Assume finally that a is a 2-vertex for C adjacent to 4 and 5. Then, since $a43x_2$ is no C_4, $ax_2 \notin E$, and since the P_4 $a43x_2$ is not missing x_1, $ax_1 \in E$ but now $1x_234ax_1$ is a C_6 - contradiction. Thus $N_{4,5} = \emptyset$. With a similar argument one shows that $N_{3,4} = \emptyset$.
Now, C has no 2-vertices. Since u has a join to each Q_i, 5-vertices are universal. Thus, since G is prime, C has no 5-vertices. Note that Q_3, Q_4 and Q_5 are modules since only the edge set between Q_1 and Q_2 is not a join; thus, $|Q_j| = 1$ for $j \in \{3,4,5\}$. □

Since G is C_4-free, $Q_1 \cup Q_2$ induces a co-bipartite chain graph. If $|Q_1| \geq 3$ or $|Q_2| \geq 3$ then $Q_1 \cup Q_2$ contains a P_4 which, together with the vertex 4 of C it would induce a co-gem in G. Thus, $|Q_1| \leq 2$ and $|Q_2| \leq 2$, and it is easy to see that in this case, G is a specific graph of the third type in Figure 6.

From now on, assume that for all $i \in \{1,\ldots,5\}$, Q_i has a join to Q_{i+1}. Assume first that at least one of the sets Q_i, $i \in \{1,\ldots,5\}$, has more than one vertex. This requires that one of the 2-vertex sets, say $N_{5,1}$, is nonempty.

Claim 5.5 *If $N_{5,1} \neq \emptyset$ then*

(i) $|Q_i| = 1$ *for* $i \in \{1,3,5\}$ *and moreover,*

(ii) $|Q_i| = 1$ *for* $i = 2$ *or* $i = 4$.

Proof. *To (i)*: Let $a \in N_{5,1}$. Since a is not a 1-vertex for the C_5 x_12345, $x_1 \in Q_1$, $N_{5,1}$ has a join to Q_1, and for the same reason, $N_{5,1}$ has a join to Q_5. Since for $x_3 \in Q_3$, $a12x_3$ is no C_4, $N_{5,1}$ has a co-join to Q_3. Thus, Q_1, Q_3 and Q_5 are modules and have size 1.

To (ii): If $a \in N_{5,1}$ is adjacent to a vertex $x_2 \in Q_2$ then, since for $x_4 \in Q_4$, ax_23x_4 is no C_4, $ax_4 \notin E$ i.e. a has a co-join to Q_4 and vice versa. Thus, the sets D_2 (D_4) of vertices in $N_{5,1}$ distinguishing Q_2 (Q_4) vertices are disjoint: $D_2 \cap D_4 = \emptyset$. Assume now that $a \in D_2$

with neighbor $x \in Q_2$ and nonneighbor $x' \in Q_2$ and $b \in D_4$ with neighbor $y \in Q_4$ and nonneighbor $y' \in Q_4$. Since $a \in D_2$, a has a co-join to Q_4, and now $ax3y'5$ is a C_5. Since $b \in D_4$, b has a co-join to Q_2. Since b is not a 1-vertex for the C_5 $ax3y'5$, $ab \in E$ but now the P_4 $x'xab$ is missing y' - contradiction. Thus at most one of the sets D_2, D_4 is nonempty, and (ii) follows. □

From now on, assume that for at most one $i \in \{1, \ldots, 5\}$, $|Q_i| > 1$, say $|Q_j| = 1$ for $j \in \{1, 3, 4, 5\}$.

Claim 5.6 *If* $|Q_2| > 1$ *then* $U = \emptyset$.

Proof. Assume that $|Q_2| > 1$ and $U \neq \emptyset$. Then, since Q_2 is no homogeneous set, there are $x, y \in Q_2$ and $a \in N_{5,1}$ such that $ax \in E$ and $ay \notin E$. Let $u \in U$. Since $axu5$ is no C_4, $au \in E$. Since u is not a universal vertex in G, there is $a' \in N_{5,1}$ such that $a'u \notin E$; in particular, $a \neq a'$ holds. Since $xu5a'$ is no C_4, $a'x \notin E$, and since $yu5a'$ is no C_4, $a'y \notin E$. Since $a'axy$ is no P_4 missing 4, $a'a \notin E$ holds but now $ax34$ is a P_4 missing a' - contradiction. Thus, the claim holds. □

Claim 5.7 *If* $|Q_2| > 1$ *then* $N_{5,1}$ *is a clique.*

Proof. Assume not; let $a, b \in N_{5,1}$ with $ab \notin E$. If for $v \in Q_2$, $av \in E$ and $bv \notin E$ then $av34$ is a P_4 missing b - contradiction. Thus, a and b have the same neighborhoods in Q_2 which means that the co-connected component in $N_{5,1}$ containing a and b is a homogeneous set. Thus, the claim follows. □

Note that again, as for the argument after Claim 5.4, G is a specific graph since $|Q_2| \leq 2$ and $|\{1\} \cup N_{5,1}| \leq 2$.

From now on, assume that *for any* C_5 C in G with vertex set $\{1, \ldots, 5\}$ and edges $\{i, i+1\}$, $i \in \{1, \ldots, 5\}$, $|Q_i| = 1$ for all $i \in \{1, \ldots, 5\}$ holds. Then, if one of the 2-vertex sets is nonempty, G corresponds to Case (ii) in Lemma 5.1.
Finally, if C has no 2-vertices then C has no 5-vertices as well, and G is a C_5. □

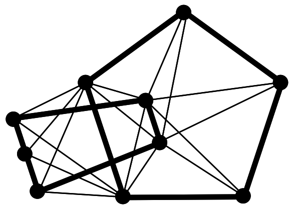

Figure 7: Two C_5's in a graph without 3-vertices

Note that in Case (ii) of Lemma 5.1, G may contain more than one C_5. Lemma 5.2 describes the relationship of them, and in particular shows that they are vertex-disjoint. To this purpose, we switch again to the complement graph.

Lemma 5.2 *Let* $G = (V, E)$ *be a graph in which for every* C_5 $C = (v_1, v_2, v_3, v_4, v_5)$*,* V *has a partition into* $\{v_1, v_2, v_3, v_4, v_5\}$*, the set* A *of 0-vertices for* C *and the set* $B = V \setminus (\{v_1, v_2, v_3, v_4, v_5\} \cup A)$ *of 3-vertices for* C *having the same neighborhoods in* C*, say,* v_2, v_4, v_5*. Then the following properties hold:*

(*i*) *The vertices v_1 and v_3 have degree 2, and the vertices in $\{v_2, v_4, v_5\} \cup B$ have degree larger than 2 (unless G is a C_5 plus possible isolated vertices).*

(*ii*) *The C_5's in G are pairwise vertex-disjoint.*

(*iii*) *If C and C' are distinct C_5's in G, A' is the set of 0-vertices for C' and $B' = V \setminus (A' \cup C')$ is the set of corresponding 3-vertices then the vertices of C of degree 2 are in A' and the other three vertices of C are in B'.*

Proof. *To* (*i*): This is obviously fulfilled.

To (*ii*): Let $C = (v_1, \ldots, v_5)$ be a C_5 in G, A its set of 0-vertices and $B = V \setminus (A \cup C)$ such that v_1 and v_3 have degree 2 and v_2, v_4 and v_5 have a join to B.
Assume first that there is another C_5 C' containing v_1 i.e. $C' = (v_1, a, b, c, d)$ with 0-vertices A' and $B' = V \setminus (A' \cup C')$. Then, since the degree of v_1 is two, $\{a, d\} = \{v_2, v_5\}$, say, $a = v_2$ and $d = v_5$. The vertex v_3 has degree two and is adjacent to v_2; thus, $v_3 \notin B'$ since B' vertices have degree at least three, i.e. $b = v_3$. Since $v_2 v_4 \notin E$, the vertex v_4, being adjacent to v_3, is not in B' i.e. $c = v_4$, and now $C' = C$.
Assume now that there is another C_5 C' containing v_2 i.e. $C' = (v_2, a, b, c, d)$ and sets A', B' as before. Then, since the degree of v_2 is larger than two, v_2 has a join to B'. Since the B' vertices have degree larger than two, the neighbors v_1 and v_3 of v_2 are not in B' i.e. $\{a, d\} = \{v_1, v_3\}$, say, $a = v_3$ and $d = v_1$. Since v_1 and v_3 have degree two, they have a co-join to B'. Now b, as a neighbor of $a = v_3$, and c, as a neighbor of $d = v_1$, are not in B' i.e. $b = v_4$ and $c = v_5$ i.e. $C' = C$.
Assume finally that there is another C_5 C' containing v_5 i.e. $C' = (v_5, a, b, c, d)$ and sets A', B' as before. Again, since the degree of v_5 is larger than two, v_5 has a join to B'. Since the B' vertices have degree larger than two, the neighbor v_1 of v_5 is not in B' i.e. $v_1 \in \{a, d\}$, say, $a = v_1$. The vertex v_1, having degree two, has a co-join to B'. Then v_2, as a neighbor of v_1, is not in B' i.e. $b = v_2$. Now the neighbor v_3 of v_2, having degree two, is not in B' i.e. $c = v_3$, and now the neighbor v_4 of v_3 is not in B' i.e. $d = v_4$ and $C' = C$.

To (*iii*): This is obviously fulfilled. □

6 Structure of (P_5,gem)-free graphs

A graph is *chordal* if it contains no induced cycles C_k, $k \geq 4$. See e.g. [15, 16, 10, 2] for properties of chordal graphs. A graph is *co-chordal* if its complement graph is chordal. A vertex v is *simplicial* in G if its neighborhood $N(v)$ in G is a clique. A vertex is *co-simplicial* in G if it is simplicial in $\overline{G}$.

Clearly co-chordal gem-free graphs are a subclass of (P_5,gem)-free graphs. In fact every (C_4,C_6,co-gem)-free graph without a C_5 is chordal, and thus co-chordal gem-free graphs are a subclass of ($2K_2$,$\overline{C_6}$,gem)-free graphs. Their structure is described in more detail in [1], and based on this structure, it is shown in [1] that the clique-width of co-chordal gem-free graphs is at most 4.
From the list of all minimal forbidden subgraphs for interval graphs given by Lekkerkerker and Boland in [12] (see Figure 8) it easily follows that a chordal co-gem-free graph G is either an interval graph or contains a 3-sun. Except the cycles C_4, C_5, C_6 and the 3-sun, all other forbidden subgraphs in Figure 8 contain a co-gem.

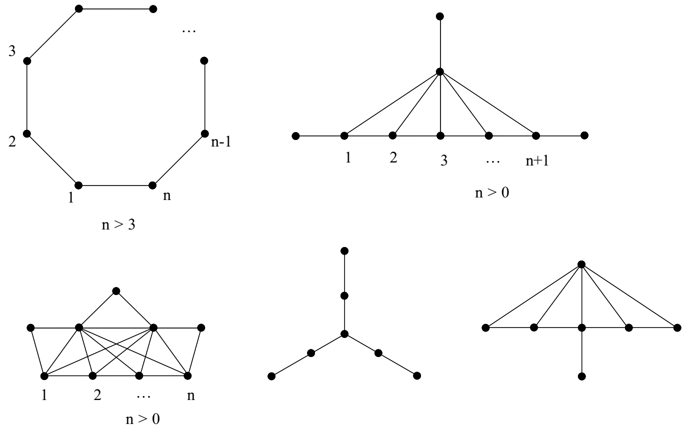

Figure 8: List of all minimal forbidden subgraphs of interval graphs

To characterize all prime ($2K_2$,$\overline{C_6}$,gem)-free graphs satisfying Case *(ii)* of Corollary 5.1, we define the following kind of substituting a C_5 into a vertex: For a graph G and a vertex v in G, let the result of the extension operation $ext(G,v)$ denote the graph G' resulting from G by replacing v with a C_5 (v_1,v_2,v_3,v_4,v_5) of new vertices such that v_2,v_4 and v_5 have the same neighborhood in G as v and v_1,v_3 have only their C_5 neighbors i.e. have degree 2 in G'. For a vertex set $U \subseteq V$ of G, let $ext(G,U)$ denote the result of applying repeatedly the extension operation to all vertices of U. Note that the resulting graph does not depend on the order of replacing U vertices.

Lemma 6.1 *If G is prime then $ext(G,v)$ is prime.*

Proof. Assume that $G' = ext(G,v)$ contains a homogeneous set H'. Then, since G is prime, H' contains either one of the degree 2 vertices v_1,v_3 or at least two of v_2,v_4,v_5. Let $A := \overline{N_G}(v)$ and $B := N_G(v)$.
Assume first that $v_1 \in H'$. If $v_2 \notin H'$ then, since v_2 is adjacent to v_1, v_2 has a join to H' i.e. $(A \cup \{v_4,v_5\}) \cap H' = \emptyset$. Now, since v_4 and v_1 are nonadjacent, v_4 has a co-join to H' i.e. $(B \cup \{v_3\}) \cap H' = \emptyset$ but now $H' = \{v_1\}$ - contradiction. Thus, $v_2 \in H'$. Since v_5 distinguishes v_1 and v_2, also $v_5 \in H'$ and similarly for v_3 and v_4. Now, since H' is a proper subset of $V(G')$, $(H' \setminus \{v_1,v_2,v_3,v_4,v_5\}) \cup \{v\}$ is a homogeneous set in G - contradiction. Thus, the degree 2 vertices v_1 and v_3 are not in H'.
Now assume that v_2 is in H'. Then also one of v_4,v_5 is in H', say v_4. Since v_1 distinguishes v_2 and v_4, v_1 must be in H' - contradiction. The same contradiction is obtained for v_5 in H'. Now assume that H' contains v_4 and v_5 but not v_2. Then again, since v_1 distinguishes v_4 and v_5, v_1 is in H' - contradiction. This shows that G' is prime. $\square$

Note that the other direction of Lemma 6.1 does not hold as the example of extending an end-vertex of a P_3 shows: the P_3 $P = (v,b,a)$ is not prime but $ext(P,v)$ is prime.

Now, based on the operation $ext(G,v)$, we define the following graph classes $\mathcal{C}_k$, $k \geq 0$, k being the number of C_5's contained in a graph $G \in \mathcal{C}_k$.

- for $k \geq 0$, let $\mathcal{C}_k$ be the class of *prime* graphs $G' = ext(G,Q)$ resulting from a (not necessarily prime) co-chordal gem-free graph G by extending a clique Q of exactly k co-simplicial vertices of G.

Thus, $\mathcal{C}_0$ is the class of prime co-chordal gem-free graphs.

Lemma 6.2 *Let G be a co-chordal gem-free graph and let Q be a clique of k co-simplicial vertices in G. Then $G' = ext(G, Q)$ is a ($2K_2$,$\overline{C_6}$,gem)-free graph containing exactly k vertex-disjoint C_5's satisfying condition (ii) of Corollary 5.1.*

Proof. We first show that extending one co-simplicial vertex x to a C_5 C_x maintains the required properties in the resulting graph G'. Then the same argument can be applied repeatedly. Let x be a co-simplicial vertex of G which is extended to a C_5 $C_x = (x_1, x_2, x_3, x_4, x_5)$ with degree 2 vertices x_1 and x_3 and x_2, x_4, x_5 having the same neighborhoods in G as x. Let $B := N_G(x)$ and $A := \overline{N_G}(x)$. Since x is co-simplicial and G is gem-free, we obtain that $G(B)$ is a cograph and A is stable. First, we show that G' is ($2K_2$,$\overline{C_6}$,gem)-free.

Claim 1. **G' is $2K_2$-free.**

Assume that G' contains a $2K_2$ H with edges ab and cd. Since G is $2K_2$-free, H contains either one of the degree 2 vertices x_1, x_3 or at least two of the vertices x_2, x_4, x_5.
If the $2K_2$ is x_1b, cd then $b \in \{x_2, x_5\}$ but then c and d are in the stable set A - contradiction. The same contradiction is obtained if x_3 is in H. Thus, H contains none of the degree 2 vertices x_1, x_3 i.e. H contains at least two of the vertices x_2, x_4, x_5. Assume first that x_2 is in H; let the $2K_2$ be x_2b, cd. Then H contains another vertex from x_4, x_5, say $c = x_4$. Thus $b \in B$ but b is also adjacent to x_4 - contradiction. Now assume that x_2 is not in H i.e. x_4 and x_5 are in H, say $H = \{x_4x_5, cd\}$. Then c and d are in A - contradiction. This shows that G' is $2K_2$-free.

Claim 2. **G' is gem-free.**

Assume that G' contains a gem H with P_4 $abcd$ and universal vertex e. Again, since G is gem-free, H contains either one of the degree 2 vertices x_1, x_3 or at least two of the vertices x_2, x_4, x_5. Note first that the degree 2 vertices x_1 and x_3 are impossible in H since be and ce are edges.
If $a = x_2$ then $e \notin \{x_4, x_5\}$. Now, $\{c, d\} \cap \{x_4, x_5\} \neq \emptyset$ but in either case, there is a gem vertex $u \notin \{x_4, x_5\}$ distinguishing x_2 and x_4 or x_2 and x_5 - contradiction. The same problem occurs if $b = x_2$. Note that $e \neq x_2$ since B is a cograph. Now x_2 is not in H, and thus x_4 and x_5 is in H. In either case, there is a vertex in H distinguishing x_4 and x_5 - contradiction.

Claim 3. **G' is $\overline{C_6}$-free.**

Assume that G' contains a $\overline{C_6}$ H with vertices a, b, c, d, e, f. Again, since G is $\overline{C_6}$-free, H contains either one of the degree 2 vertices x_1, x_3 or at least two of the vertices x_2, x_4, x_5.
Note first that the degree 2 vertices x_1 and x_3 are impossible in H. If x_2 is in H then H contains at least one of x_4, x_5. Now, if H contains only one of them, say x_4, then H contains a vertex distinguishing x_2 and x_4, and if H contains both of them then H contains a vertex distinguishing x_4 and x_5 - contradiction.

It is easy to see that $G' = ext(G, Q)$ contains exactly k vertex-disjoint C_5's having the partition property of Corollary 5.1 (ii). □

The inverse operation to extending co-simplicial vertices by a C_5 is contracting every C_5 C to a single vertex x_C representing the C_5 C. Assume that C has exactly two vertices, say x_1 and x_3, of degree 2, and the other three vertices x_2, x_4, x_5 inducing a $K_1 + K_2$ are a homogeneous set in $G(V \setminus \{x_1, x_3\})$. Then pick one of the vertices x_2, x_4, x_5, say x_2, as representative vertex x_C for C. Let $contr(G)$ denote the graph resulting from contraction of all C_5's in G.

Lemma 6.3 *Let G be a ($2K_2$,$\overline{C_6}$,gem)-free graph containing exactly k vertex-disjoint C_5's having the partition property of Corollary 5.1 (ii). Then $contr(G)$ is a co-chordal gem-free graph and for every C_5 C in G, the representative vertex x_C is co-simplicial in $contr(G)$ and forms a clique with the other $k-1$ representative vertices of all other C_5's.*

Proof. Note first that contr(G) is an induced subgraph of G and thus, contr(G) is ($2K_2$,$\overline{C_6}$,gem)-free. Moreover, it is C_5-free and thus co-chordal since C_5's in G are vertex-disjoint. The fact that for every C_5 C in G, the representative vertex x_C is co-simplicial in contr(G), is obvious since the set A of nonneighbors of x_C in contr(G) is stable. The set of representative vertices forms a clique in contr(G) since by Lemma 5.2, (iii), for C_5's C and C', the representative vertices x_C and $x_{C'}$ are adjacent. □

Lemma 6.4 *Let G be a prime graph. Then*

- (i) *G is co-chordal gem-free if and only if $G \in \mathcal{C}_0$, and*
- (ii) *for $k \geq 1$, G is a ($2K_2$, $\overline{C_6}$,gem)-free graph with exactly k vertex-disjoint C_5's satisfying condition (ii) of Corollary 5.1 if and only if $G \in \mathcal{C}_k$.*

Proof. *To (i):* By definition, a prime graph G is co-chordal gem-free if and only if $G \in \mathcal{C}_0$.

To (ii): First, let G be a prime ($2K_2$, $\overline{C_6}$,gem)-free graph satisfying condition (ii) of Corollary 5.1. Then all the C_5's of G are vertex-disjoint. By Lemma 6.3, the graph contr(G) is co-chordal gem-free and moreover, for every C_5 C in G, the representative vertex x_C is co-simplicial in contr(G) and forms a clique Q with all other representative vertices. Now by Lemma 6.2, G is the extension ext($contr(G), Q$) of the co-chordal gem-free graph contr(G) and thus, as a prime graph, belongs to $\mathcal{C}_k$.
Now let $G \in \mathcal{C}_k$. Then by definition, $G = ext(G', Q)$ for a co-chordal gem-free graph G' by extending a clique Q of k co-simplicial vertices of G. This means by Lemma 6.2 that G is ($2K_2$, $\overline{C_6}$,gem)-free satisfying condition (ii) of Corollary 5.1. □

Recall that a graph is specific if it is one of the graphs in Figure 6 or one of its induced subgraphs. Lemma 3.1, 4.1, 5.1 and 6.4 imply the following:

Theorem 1 (Structure Theorem) *A connected and co-connected graph G is (P_5,gem)-free if and only if the following conditions hold:*

(1) *The homogeneous sets of G are P_4-free;*

(2) *For the characteristic graph G^* of G, one of the following conditions holds:*

- (2.1) *G^* is a matched co-bipartite graph;*
- (2.2) *$\overline{G^*}$ is a specific graph;*
- (2.3) *there is a $k \geq 0$ such that G^* is in $\mathcal{C}_k$.*

Proof. 1. $\Longrightarrow$: If the graph G is connected and gem-free then obviously the homogeneous sets are P_4-free. Moreover, due to Lemma 3.1, if G^* contains a $2K_2$ then G^* is a matched co-bipartite graph. Due to Lemma 4.1, if $\overline{G^*}$ is (C_4,co-gem)-free containing a C_6 then it is a specific graph. Due to Lemma 5.1 and 6.4, if $\overline{G^*}$ is (C_4,C_6,co-gem)-free containing a C_5 then it fulfills condition (2.2) or (2.3). In the final case, $\overline{G^*}$ is a co-gem-free chordal graph and thus G^* fulfills (2.3).

2. $\Longleftarrow$: Assume now that the homogeneous sets of G are cographs, and G^* is of one of the types (2.1), (2.2), (2.3). In case (2.1) the graph G^* is (P_5,gem)-free by Lemma 3.1. In case (2.2) each of the graphs in Figure 6 is (house,co-gem)-free, and thus each complement of a specific graph is (P_5,gem)-free. In case (2.3), Lemma 6.4 implies that each prime graph $G^* \in \bigcup_{k=0}^{\infty} \mathcal{C}_k$ is ($2K_2$,$\overline{C_6}$,gem)-free, and thus it is (P_5,gem)-free.

Thus it remains to show that by substituting vertices by cographs, no P_5 or gem is created. To this purpose we need the fact that for a homogeneous set H and a P_4 $abcd$, at most one of the vertices a, b, c, d is in H. Now assume that G contains a P_5. This means that after shrinking H to one vertex, the P_5 in G is still a P_5 in G^* - contradiction. Analogously, a contradiction is obtained if G contains a gem. This means that G is indeed (P_5,gem)-free. $\square$

In [1], it is shown that the clique-width of co-chordal gem-free graphs is at most 4. Now, based on Theorem 1, it is not hard to show that the clique-width of (P_5,gem)-free graphs is at most 5.

Acknowledgement. The authors would like to thank Raffaele Mosca (Pescara, Italy) and Michaël Rao (Metz, France) for helpful discussions.

References

[1] A. Brandstädt, H.-O. Le, R. Mosca, Chordal co-gem-free graphs have bounded clique width, *Manuscript* 2002

[2] A. Brandstädt, V.B. Le, J. Spinrad, Graph Classes: A Survey, *SIAM Monographs on Discrete Math. Appl., Vol.* 3, SIAM, Philadelphia (1999)

[3] D.G. Corneil, H. Lerchs, L. Stewart-Burlingham, Complement reducible graphs, *Discrete Applied Math.* 3 (1981) 163-174

[4] D.G. Corneil, Y. Perl, L.K. Stewart, Cographs: recognition, applications, and algorithms, *Congressus Numer.* 43 (1984) 249-258

[5] D.G. Corneil, Y. Perl, L.K. Stewart, A linear recognition algorithm for cographs, *SIAM J. Computing* 14 (1985) 926-934

[6] B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraph grammars, *J. Comput. Syst. Sciences*, 46 (1993) 218-270

[7] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique width, *Theory of Computing Systems* 33 (2000) 125-150

[8] A. Cournier, M. Habib, A new linear algorithm for modular decomposition, *LIRMM, University Montpellier* (1995), Preliminary version in: *Trees in Algebra and Programming - CAAP '94*, LNCS 787 (1994) 68-84

[9] E. Dahlhaus, J. Gustedt, R.M. McConnell, Efficient and practical algorithms for sequential modular decomposition, *J. Algorithms* 41 (2001) 360-387

[10] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, *Academic Press New York* (1980)

[11] C.T. Hoàng, B. Reed, Some classes of perfectly orderable graphs, *J. Graph Theory* 13 (1989) 445-463

[12] C.G. Lekkerkerker, J.Ch. Boland, Representation of finite graphs by a set of intervals on the real line, *Fund. Math.* 51 (1962) 45-64

[13] R.M. McConnell, J. Spinrad, Modular decomposition and transitive orientation, *Discrete Math.* 201 (1999) 189-241

[14] R. Mosca, A note on maximum stable sets in certain P_5-free graphs, accepted for *Discrete Applied Math.*

[15] D.J. Rose, R.E. Tarjan, G.S. Lueker, Algorithmic aspects of vertex elimination on graphs, *SIAM J. Comput.* 5 (1976) 266-283

[16] R.E. Tarjan, M. Yannakakis, Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs, *SIAM J. Comput.* 13 (1984), 566-579; Addendum: *SIAM J. Comput.* 14 (1985), 254-255

[17] M. Yannakakis, The complexity of the partial order dimension problem, *SIAM J. Algebraic and Discrete Methods* 3 (1982) 351-358