On a property of minimal triangulations

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Abstract

A graph H has the property MT, if for all graphs G, G is H-free if and only if every minimal (chordal) triangulation of G is H-free. We show that a graph H fulfils property MT if and only if H is edgeless, H is connected and is an induced subgraph of P_5, or H has two connected components and is an induced subgraph of $2P_3$.

This completes results of Parra and Scheffler, who have shown that MT holds for $H = P_k$, the path on k vertices, if and only if $k \leqslant 5$ [11], and of Meister, who proved that MT holds for ℓP_2, ℓ copies of a P_2, if and only if $\ell \leqslant 2$ [9].

1 Introduction

Minimal triangulations of graphs have been studied for various reasons. The first $O(nm)$ time algorithm to compute a minimal triangulation of a graph dates back to 1976 [12]. A lot of the interest in minimal triangulations of graphs stems from their strong relation to the treewidth and the minimum fill-in of graphs. Research on the algorithmic complexity of those two well-known and fundamental NP-hard graph problems on particular graph classes, to a large extent carried out in the last decade of the last century, has led to interesting insights on the structure of minimal triangulations of graphs and how to use them for the design of efficient algorithms to compute the treewidth and the minimum fill-in on particular graph classes (see e.g. [2, 3, 4, 8, 11]). An excellent survey on the research about minimal triangulations has been provided by Heggernes [7] in a special issue of *Discrete Mathematics* entitled "Minimal separation and minimal triangulation" [1].

One of the early and important contributions concerning minimal triangulations of graphs in particular graph classes was given by Möhring. He showed that every minimal triangulation of an AT-free graph is an AT-free graph, and thus an interval graph [10]. It was soon discovered that indeed, a graph G is AT-free if and only if every minimal triangulation of G is AT-free (see e.g. [11]). In this line of research, Parra and Scheffler showed that for all $k \leqslant 5$, and every graph G, G is P_k-free if and only if every minimal triangulation of G is P_k-free, where P_k is the path on k vertices [11]. They also showed by means of an example that this property does not hold if $k \geqslant 6$ [11]. Recently Meister reconsidered this property and showed among others, that for every graph G, G is $2K_2$-free if and only if every minimal triangulation of G is $2K_2$-free, where $2K_2 = 2P_2$ is the union of two copies of P_2 [9].

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Here we continue this line of research. In fact, we give a characterisation (see Theorem 8) that completely settles the question which graphs H have the following property that we call property MT: *For every graph G, G is H-free if and only if all minimal triangulations of G are H-free.*

2 Preliminaries

Throughout this paper let $G = (V, E)$ be a finite simple undirected graph. The neighbourhood $N(v)$ of a vertex $v \in V$ is the set of vertices u adjacent to v. For a vertex set $S \subseteq V$, its neighbourhood is defined by $N(S) = \{u \in V \setminus S \mid N(u) \cap S \neq \varnothing\}$. The subgraph of G induced by a vertex set $S \subseteq V$ is denoted by $G[S]$. We denote the vertex set of a (connected) component of a graph by C. The corresponding maximal connected induced subgraph of G is denoted by $G[C]$. The number of connected components of a graph G is denoted by $c(G)$.

Let $H_1 + H_2$ denote the union of disjoint copies of H_1 and H_2. For $\ell \geqslant 1$, we denote by ℓG the disjoint union of ℓ copies of G. The graph P_r is an induced path on r vertices and the K_r is a complete graph on r vertices. A set $S \subseteq V$ is a clique (resp. independent set) of a graph $G = (V, E)$ if for all $u, v \in S$ the vertices u and v are adjacent (resp. non-adjacent). The maximum cardinality of an independent set of G is denoted by $\alpha(G)$.

A graph $G = (V, E)$ is *chordal* if each cycle of length at least four has a chord in G. Chordal graphs are intersection graphs of subtrees of a tree. The corresponding intersection model of a chordal graph is called a *clique tree*. The vertex set of such a clique tree T of a graph $G = (V, E)$ is the set of all maximal cliques of G and for every $v \in V$ the set of all nodes of the tree containing v is connected. For more details on chordal graphs and other graph classes we refer to [5, 6].

The fundamental concepts of our paper are minimal triangulations and minimal separators. Both concepts are strongly related to chordal graphs. A *triangulation* of a graph G is a chordal graph $G^* = (V, F)$ with $E \subseteq F$. The edges in $F \setminus E$ are called *fill-edges*. G^* is a *minimal triangulation* of G if there is no triangulation of G whose fill-edges form a proper subset of $F \setminus E$. A set $S \subseteq V$ is a *separator* of G if $G - S$ is disconnected. For two vertices $u, v \in V$ of G, a separator $S \subseteq V$ is a *u, v-separator*, if u and v belong to different components of $G - S$, and S is a *minimal u, v-separator* if no proper subset S' of S is a u, v-separator. Finally S is a *minimal separator* of G if there are vertices $u, v \in V$ such that S is a minimal u, v-separator of G. We denote the set of minimal separators of a graph G by Δ_G. Note that there may be minimal separators of a graph G such that one is a proper subset of the other. We mention a well-known characterization of chordal graphs as precisely those graphs for which each minimal separator is a clique.

We shall need some deeper understanding of minimal separators and their relations to minimal triangulations. Let $S \subseteq V$ be a separator of G, and let $G[C]$ be a connected component of $G - S$. Then, clearly $N(C) \subseteq S$. If $N(C) = S$ then $G[C]$ is a called an *S-full* component of G. It is well-known that S is a minimal separator of G if G has at least two S-full components (see e.g. [6]).

Let S and T be two minimal separators. S and T are *parallel*, symbolically $S \| T$, if there is a component of $G - S$ containing all vertices of $T \setminus S$. Otherwise S and T *cross*, also written as $S \# T$. Notice that both $\|$ and $\#$ are symmetric relations.

Sets of pairwise parallel minimal separators and minimal triangulations are closely related which is nicely illustrated by the fundamental theorem of Parra and Scheffler. To formulate it we need some more notation. Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by completing S, i.e., by adding an edge between every pair of non-adjacent vertices of S. If $\Gamma \subseteq \Delta_G$ then G_Γ denotes the graph obtained by completing all the minimal separators of Γ.

Theorem 1 ([11])**.** *Let $\Gamma \subseteq \Delta_G$ be a maximal set of pairwise parallel minimal separators of G. Then $G^* = G_\Gamma$ is a minimal triangulation of G and $\Delta_{G^*} = \Gamma$. Conversely, let G^* be a minimal triangulation of a graph G. Then Δ_{G^*} is a maximal set of pairwise parallel minimal separators of G and $G^* = G_{\Delta_{G^*}}$.*

We shall also need the following fact. Let G^* be a minimal triangulation of G. By Theorem 1, every minimal separator of G^* is also a minimal separator of G. Moreover, $G[C]$ is a connected component of $G - S$ if and only if $G^*[C]$ is a connected component of $G^* - S$.

For details and references to original work we point to [7].

Finally let us consider some simple cases. It is not hard to see that only chordal graphs H fulfil property MT: for a non-chordal H, the graph $G = H$ is not H-free while all minimal triangulations of G are chordal and thus H-free. More interesting, one direction of MT is widely known to hold for all chordal graphs H, see for instance [11]:

Lemma 2. *Let H be an induced chordal subgraph of G. Then there is a minimal triangulation G^* of G preserving H.*

Proof. Let $G = (V, E)$, $B \subseteq V$, $A = V \setminus B$ and $H = G[B]$ a chordal graph. Consider the graph $G' = (V, E')$ with $E' = E \cup \{av : a \in A, v \in V\}$. To see that the graph G' is chordal take a clique tree T of H and add to each maximal clique of H assigned to a node of T all vertices of A. This produces a clique tree of G' and thus G' is chordal. Consequently G' is a triangulation of G and $G'[B] = H$. Finally there is a minimal triangulation $G^* = (V, F)$ of G such that $E \subseteq F \subseteq E'$ and $G^*[B] = H$. □

This implies that for chordal graphs H, and only those are of interest, property MT is equivalent to the following property MT′: *For every H-free graph G, all minimal triangulations of G are H-free.*

Before we completely characterise all graphs H having property MT (or, equivalently, MT′), let us recall the results known prior to our work. Parra and Scheffler have shown that MT holds for $H = P_k$, the path on k vertices, if and only if $k \leqslant 5$ [11]. In Figure 2(c) we give their counterexample for the P_6. Meister proved that MT holds for ℓP_2 if and only if $\ell \leqslant 2$ [9]. We extend his result in Lemma 4 and slightly modify his counterexample, originally for the $3P_2$, in Figure 2(e).

3 Graphs that fulfil property MT

Lemma 3. *Property MT holds for all edgeless graphs H.*

Proof. Let H be an edgeless graph on k vertices. Then G is H-free if and only if $\alpha(G) < k$. Since a triangulation G^* of G is obtained by adding edges to G, we have $\alpha(G^*) \leqslant \alpha(G)$ for all minimal triangulations G^* of G. Hence, for each edgeless graph H, if G is H-free then every minimal triangulation of G is H-free. □

Lemma 4. *Property MT holds for $H = 2P_3$.*

Proof. Since $2P_3$ is chordal, Lemma 2 implies that we only have to prove MT′: for every $2P_3$-free G, all minimal triangulations of G are $2P_3$-free. To this end, we consider a graph G and one of its minimal triangulations G^* such that $H = 2P_3$ is an induced subgraph of G^*, and we show that G contains a $2P_3$ as induced subgraph as well. Notice that each minimal separator S of the chordal graph G^* is a clique, and thus S cannot contain two non-adjacent vertices of G^*.

Let $G = (V, E)$, $B \subseteq V$, $A = V \setminus B$ and $H = G^*[B]$ a chordal graph. Let (u, v, w) and (x, y, z) be two paths in G^* inducing $H = 2P_3$. If none of the edges of $G^*[B]$ is a fill-edge then H is an induced subgraph of G, we are done.

Case 1: Assume that only one path contains fill-edges, say (u, v, w) and let uv be one of them. By Theorem 1 there is a minimal separator S of G with $S \cap B = \{u, v\}$. Since there are no fill-edges in (x, y, z), the vertices x, y and z belong to one connected component $G[C]$ of $G - S$. No matter whether $G[C]$ is an S-full component or not, there is another S-full component $G[C']$ of G. Since S is a minimal separator of G, and C' is full, there is a u–v-path in $G[C' \cup S]$. We consider a shortest one. It contains a P_3-subpath since $uv \notin E$. This subpath together with (x, y, z) form a $2P_3$ in G, see Figure 1.

Case 2: Assume that there are fill-edges in both paths, w.l.o.g. let uv and xy be fill-edges in G^* w.r.t. G. Similar to Case 1, there is a minimal separator S of G with $S \cap B = \{u, v\}$ and a minimal separator T of G with $T \cap B = \{x, y\}$. By Theorem 1, the minimal triangulation G^* is obtained from G by completing all minimal separators of Δ_{G^*}, and all of them are pairwise parallel in G, hence S and T are parallel in G. Therefore, there is a connected component $G[C]$ of $G - S$ containing all vertices of $T \setminus S \supseteq \{x, y\}$. Since $G - S$ has at least two S-full components, there is an S-full component $G[C']$ of G such that $C \neq C'$. Note that $T \cap C' = \varnothing$ since T and S are parallel. Now consider the graph $G - T$. This graph has a connected component D containing all vertices of $C' \cup \{u, v\}$, and there is also a T-full component $D' \neq D$.

Figure 1: Each curve denotes path of length $\geqslant 0$, and each (dotted) straight line denotes a (fill-)edge.

Now similar to the first case, we take a shortest u–v-path of G with all interior vertices in C', and a shortest x–y-path of G with all interior vertices in D'. Each contains a P_3-subpath, and those two form a $2P_3$ in G, see Figure 1. □

Similarly one can show the following lemma.

Lemma 5. *Property MT holds for $H = P_3 + P_2$, $H = P_3 + P_1$ and $H = P_2 + P_1$.*

Thus all induced subgraphs of $2P_3$ with (at most) two connected components fulfil property MT.

4 Graphs that do not fulfil property MT

In Figure 2 we provide counterexamples: the given graph G shows that the graph H (mentioned in the caption) does not fulfil property MT, since G is H-free and the minimal triangulation G^* of G, obtained by adding the dotted fill-edge, contains H as induced subgraph.

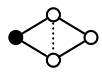

Figure 2(a): $H = C_3$

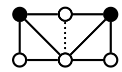

Figure 2(b): $H = K_{1,3}$

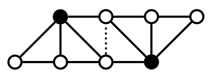

Figure 2(c): $H = P_6$

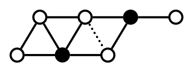

Figure 2(d): $H = P_4 + P_1$

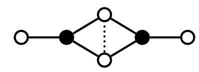

Figure 2(e): $H = P_2 + 2P_1$

Figure 2: Counterexamples with vertices in A in black and one dotted fill-edge. The graph in (c) appeared in [11] and the example in (e) was adopted from [9].

We denote by $\mathcal{F}_{\mathrm{MT}}$ the set of graphs H not having property MT that are given in Figure 2. Observe that the provided counterexamples have a special structure: Each graph $G = (V, E)$ contains exactly one chordless cycle of length four and no other chordless cycle of length at least four. By Theorem 1, these graphs have exactly two minimal triangulations. One of them is the graph G^* obtained by adding the dotted fill-edge e; and G^* contains H. As usual let $H = G^*[B]$ and $A = V \setminus B$. That is, $G - A = H - e$. Note that in all counterexamples the set A consists of a single vertex or contains exactly two vertices, which are non-adjacent. This is crucial for the proof of Theorem 7, that heavily relies on the following lemma.

Lemma 6. *Let $\mathcal{F}_{\mathrm{MT}} = \{C_3, K_{1,3}, P_6, P_4 + P_1, P_2 + 2P_1\}$. Let H' be a chordal graph and $Q' \in \mathcal{F}_{\mathrm{MT}}$ such that Q' is a proper induced subgraph of H' with $c(Q') \leqslant c(H')$. Then there is a simplicial vertex x of H' and a graph $Q \in \mathcal{F}_{\mathrm{MT}}$ such that Q is an induced subgraph of $H' - x$.*

Proof. We distinguish between three cases. First let H' be a complete graph. Then we choose any $Q = C_3$ being an induced subgraph of H'. Clearly all vertices outside $Q' = Q$ are simplicial in H'.

Next let H' be a non-complete graph that contains a cycle. Since H' is chordal it contains a C_3. Let Q be an arbitrary C_3 in H'. Since H' is chordal and not complete, it contains two non-adjacent simplicial vertices [6]. At most one of them belongs to Q. Therefore we can choose x as desired.

Finally let H' be an acyclic graph, that is, a forest. Then the proper induced subgraph Q' is a forest too. Now we choose $Q = Q'$ and x such that x is a leaf of H' that does not belong to Q. It remains to show that such an x always exist. Otherwise Q' would contain all leaves of H'. Since $c(Q') \leqslant c(H')$, this would imply $H' = Q'$, a contradiction. □

Theorem 7. *If a chordal graph H contains any of the graphs $C_3, K_{1,3}, P_6, P_4 + P_1, P_2 + 2P_1$ as induced subgraph Q and $c(Q) \leqslant c(H)$ then H does not fulfil property MT.*

Proof. We prove the following

Claim: *If a chordal graph H contains any of the graphs $C_3, K_{1,3}, P_6, P_4 + P_1, P_2 + 2P_1$ as induced subgraph Q and $c(Q) \leqslant c(H)$ then there is an H-free graph G and a minimal* *triangulation G^* of G such that G^* contains H as induced subgraph and has exactly one fill-edge, $H = G^*[B]$ and $A = V \setminus B$ is an independent set of size at most two.*

The proof is by induction on $\subset$, the well-founded relation "proper induced subgraph".

As pointed out earlier in this section, the chordal graphs $C_3, K_{1,3}, P_6, P_4 + P_1, P_2 + 2P_1$ violate MT, see Figure 2, and have a counterexample G and G^* as stated in the Claim. Thus these five graphs provide the base step of the induction.

For the inductive step let H' be a chordal graph having a graph $Q' \in \mathcal{F}_{\mathrm{MT}}$ as a proper induced subgraph such that $c(Q') \leqslant c(H')$. By Lemma 6, there is a vertex x such that x is simplicial in H' and $H = H' - x$ is a chordal graph having any of those five graphs, say Q, as induced subgraph. By the assumption of our induction the Claim is fulfilled for H. Hence there is a counterexample $G = (V, E)$ and G^* such that G^* is a minimal triangulation of G with exactly one fill-edge, $H = G^*[B]$ and $A = V \setminus B$ is an independent set of size at most two. Furthermore, let $b_1 b_2$ be the fill-edge of G^*.

Now we construct a graph G' and a minimal triangulation G'^* of G', sets A' and B' having the properties of the Claim with respect to H':

1. Take a simplicial vertex x of H' that does not belong to some proper induced subgraph $Q \in \mathcal{F}_{\mathrm{MT}}$ of H' with $c(Q) \leqslant c(H')$. By Lemma 6, such a vertex x exists.
2. To obtain G'^*, add x to G^* and make x adjacent to all vertices of $N_{H'}(x) \subseteq V$.
3. $G' = G'^* - b_1 b_2$, $B' = B \cup \{x\}$ and $A' = A$.

Now we show that G', G'^*, B' and A' is indeed a counterexample for H' with the properties of the Claim. The graph G'^* is chordal, since $G^* = G'^* - x$ is chordal and x is simplicial in H', and thus by construction also in G'^*. Clearly $G'^*[B \cup \{x\}] = H'$ and A' is an independent set of size at most two. Furthermore G' is not chordal since $G = G' - x$ and G'^* has only one fill-edge and is thus a minimal triangulation of G'. Finally H' is not an induced subgraph of G' since otherwise H would be an induced subgraph of G. This completes the proof. □

5 Characterisation

The following theorem characterises the class $\mathcal{C}_{\mathrm{MT}}$ of all those graphs H having property MT by combining lemmas and theorems of the previous sections.

Theorem 8. *A graph H fulfils property MT if and only if H is edgeless, H is connected and is an induced subgraph of P_5, or H has two connected components and is an induced subgraph of $2P_3$.*

Proof. We observed in Section 2 that all graphs in $\mathcal{C}_{\mathrm{MT}}$ are chordal.

Consider connected graphs H. By Theorem 7, a connected graph $H \in \mathcal{C}_{\mathrm{MT}}$ does not contain C_3 as induced subgraph, and all connected graphs in $\mathcal{C}_{\mathrm{MT}}$ are trees. The forbidden $K_{1,3}$ implies that each tree is a path. Because no connected graph in $\mathcal{C}_{\mathrm{MT}}$ contains P_6, each path in $\mathcal{C}_{\mathrm{MT}}$ has at most five vertices. Finally, it was shown in [11] that all induced subgraphs of P_5 fulfil MT.

Consider graphs H with $c(H) = 2$. By Theorem 7, a graph in $\mathcal{C}_{\mathrm{MT}}$ with exactly two connected components does not contain $P_4 + P_1$; thus it could only be an induced subgraph of $2P_3$. On the other hand, $P_2 + P_2$ fulfils MT [9], and Lemmas 4 and 5 show that all other induced subgraph H of $2P_3$ with $c(H) = 2$ fulfil MT.

Finally if $H \in \mathcal{C}_{\mathrm{MT}}$ has at least three connected components it is an ℓP_1 since H cannot have $P_2 + 2P_1$ as induced subgraph. $\square$

Finally together with Theorem 8, the inductive proof of Theorem 7 implies that for every graph H not fulfilling MT there is a "small counterexample" $G = (V, E)$ that has exactly two minimal triangulations. Both have only one fill-edge, and one minimal triangulation is G^* such that $H = G^*[B]$ and $A = V \setminus B$ is an independent set of G of size at most two.

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