Lower Bounds on Exponential Time Algorithms

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Outline

1 Branch & Reduce Algorithms
   - Branch & Reduce
   - Search Trees
   - Analysis

2 Lower Bounds
   - A Simple Independent Set Algorithm
   - The algorithm of Tarjan and Trojanowski
   - An Independent Set Algorithm of Beigel
   - The Set Cover and Domination Algorithm of FGK

3 Conclusions
Techniques to design exact exponential-time algorithms:
- Enumeration, Dynamic Programming, Branch & Reduce etc.

Branch & Reduce algorithms
(also called backtracking or search tree algorithms):
recursively applied to problem instances using Branching rules and Reduction rules.

- **Branching rules**: solving the problem by recursively solving smaller instances
- **Reduction rules**:
  - simplify the instance
  - (typically) reduce the size of the instance
Search Trees

- **Search Tree**: used to illustrate and analyse an execution of a *Branch & Reduce* algorithm

  - **nodes**: assigns to each node a solved problem instance
  - **root**: assigns the input to the root
  - **child**: each instance (subproblem) reached by a branching rule is assigned to a child (of the node of the original problem)
A search tree
Analysis

- Analysing Branch & Reduce algorithms:
  - Correctness and (Worst Case) Running Time

- Analysis of the Running Time:
  - To obtain an Upper Bound on the maximum number of nodes of the search tree (for an input of size \(n\)):
    1. Define a Measure for a problem instance.
    2. Lower bound the progress made by the algorithm at each branching step.
    3. Compute the collection of recurrences for all branching and reduction rules.
    4. Solve all those recurrences (to obtain a running time of the form \(O(\alpha_i^n)\) for each).
    5. Take the worst case over all solutions.

See the surveys [Woeginger, Fomin et al.].
Analysis

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- **Analysis of the Running Time**: To obtain an **Upper Bound** on the maximum number of nodes of the search tree (for an input of size $n$):
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6/47
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Lower Bounds

1. Branch & Reduce Algorithms
   - Branch & Reduce
   - Search Trees
   - Analysis

2. Lower Bounds
   - A Simple Independent Set Algorithm
   - The algorithm of Tarjan and Trojanowski
   - An Independent Set Algorithm of Beigel
   - The Set Cover and Domination Algorithm of FGK

3. Conclusions
Lower Bounds

- of the computational complexity of problems,
- for solving a particular problem by any algorithm of a class of algorithms,
- of the worst case running time of a particular algorithm.

Known Results [Alekhnovich et al., Pudlak and Impaglazzio]

Exponential lower bounds for Davis-Putnam type algorithms (DPLL) on k-SAT and SAT
Lower Bounds

- of the computational complexity of problems,
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Exponential lower bounds for Davis-Putnam type algorithms (DPLL) on k-SAT and SAT
Why Study Lower Bounds?

- **Upper bounds** on worst case running time of Branch & Reduce algorithms seem to overestimate the running time.

- **Lower bounds** on worst case running time can give an idea how far current analysis of an algorithm is from being tight.

- **Large gaps** between lower and upper bounds for some important B&R algorithms.

- Study of lower bounds leads to new insights on the particular algorithm.
**The simple Maximum Independent Set Algorithm**

**Definition (Independent Set)**
Let $G = (V, E)$ be a graph. A subset $I \subseteq V$ of vertices of $G$ is an **independent set** of $G$ if no two vertices in $I$ are adjacent.

**Definition (Maximum Independent Set (MIS))**
Given a graph $G = (V, E)$, compute an **maximum independent set** of $G$. 
Definition (Independent Set)
Let $G = (V, E)$ be a graph. A subset $I \subseteq V$ of vertices of $G$ is an independent set of $G$ if no two vertices in $I$ are adjacent.

Definition (Maximum Independent Set (MIS))
Given a graph $G = (V, E)$, compute an maximum independent set of $G$. 
The Algorithm $\text{sis}$

1. \textbf{int} $\text{sis}(G = (V, E))$ \{ 
2. \hspace{1em} \textbf{if}($|V| = 0$) \textbf{return} 0; 
3. \hspace{1em} \text{choose a vertex } v \text{ of minimum degree in } G 
4. \hspace{1em} \textbf{return} 1 + \max\{\text{sis}(G - N[y]) : y \in N[v]\}; 
5. \}
Analysis of Running Time (Upper Bound)

- **Classical Analysis** and **Standard Measure**
  (i.e. the measure of the input size is the number of vertices of the input graph).

- **Recurrence**:

  \[
  T(n) \leq (d + 1) \cdot T(n - d - 1),
  \]

  where \( d \) is the degree of the chosen vertex \( v \).

- **Solution** of recurrence: \( O((d + 1)^{n/(d+1)}) \), being maximum for \( d = 2 \),

- **Running time** of sis: \( O(3^{n/3}) \).

Tight Upper Bound!
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- **Running time** of \( s \) is: \( O(3^{n/3}) \).

**Tight Upper Bound!**
A Tight Lower Bound for Algorithm sis

**Theorem**

The algorithm sis computes a maximum independent set in time $\Omega(3^{n/3})$.

- Lower bound graph $G_k$: disjoint union of $k$ triangles.
- Algorithm sis applied to $G_k$: chooses a vertex of any triangle, branches into three subproblems $G_{k-1}$; (by removing a triangle from $G_k$)
- Search tree has $3^k = 3^{n/3}$ leaves;
The Algorithm \( tt \) of Tarjan and Trojanowski

- **Algorithm \( tt \):**
  - Branch & Reduce algorithm to compute a maximum independent set of a graph.
  - Published in 1977
  - Lengthy and tedious case analysis

- Classical analysis using standard measure: running time \( O(2^{n/3}) \).

- More precisely: authors analysis establishes \( O(2^{0.3289n}) \).
Important Properties of \( tt \)

**Minimum Degree at most 4**

If the minimum degree of the problem instance \( G \) is at most 4 then algorithm \( tt \) runs through plenty of cases.

**Minimum Degree at least 5**

Either \( G \) is 5-regular or algorithm \( tt \) “chooses ANY vertex \( w \) of degree at least 6 and branches to \( G - N[w] \) (select \( w \)) and \( G - w \) (discard \( w \))”.
Important Properties of $\text{tt}$

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Lower bound graphs of minimum degree 6!
Lower Bound Graphs

- **LB graphs**: For all positive integers $n$, $G_n = (\{1, 2, \ldots, n\}, E_6)$, where
  \[
  \{i, j\} \in E_6 \iff |i - j| \leq 6.
  \]

- **Tie break**: For graphs of minimum degree 6, the algorithm chooses smallest (resp. leftmost) vertex for branching.

- **Branching**: "select[i]" removes $i, i + 1, \ldots i + 6$; "discard[i]" removes $i$; thus on $G_n$ branches to $G_{n-7}$ and $G_{n-1}$. 
Branching
An Almost Tight Lower Bound

Definition
Let $T(n)$ be the number of leaves in the search tree obtained when executing algorithm $\mathcal{tt}$ on input graph $G_n$ using the specified tie break rules.

Recurrence

$$T(n) = T(n-7) + T(n-1)$$

Lower Bound of $\mathcal{tt}$
The running time of algorithm $\mathcal{tt}$ is $\Omega(2^{0.328173n})$. 
### Definition

Let $T(n)$ be the number of leaves in the search tree obtained when executing algorithm $tt$ on input graph $G_n$ using the specified tie break rules.

### Recurrence

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### Lower Bound of $tt$

The running time of algorithm $tt$ is $\Omega(2^{0.328173n})$. 

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**An Almost Tight Lower Bound**

**Branch & Reduce Algorithms**

**Lower Bounds**

**Conclusions**

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REMINDER : Upper Bound $O(2^{0.3289n})$. 
Why Not a Maximum Degree Rule?

Minimum Degree at least 5

If the minimum degree of $G$ is at least 5 the algorithm “chooses a vertex $w$ of maximum degree and branches to $G - N[w]$ (select $w$) and $G - w$ (discard $w$)”.

- No improvement of upper bound in classical analysis.
- Destroys our lower bound arguments.
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Find upper and lower bounds for new algorithm!
The Independent Set Algorithm bei of Beigel

- Algorithm bei:
  - Branch & Reduce algorithm to compute a maximum independent set of a graph.
  - SODA 1999
  - uses Beigel’s algorithm for sparse graphs

- claims running time $O(2^{0.303n})$.

- no analysis of upper bound available
Important Rules of be\textit{i}

**Sparse Graph Rule**

If the problem instance $G$ has at most $5n/2$ edges then algorithm be\textit{i} uses Beigel’s algorithm SparseFindMIS.

**Domination Rule**

If two adjacent vertices have comparable (by set inclusion) closed neighbourhoods then remove the (resp. one) vertex with largest neighbourhood.
Important Rules of \textit{bei}

**Sparse Graph Rule**

If the problem instance $G$ has at most $5n/2$ edges then algorithm \textit{bei} uses Beigel’s algorithm \text{SparseFindMIS}.

**Domination Rule**

If two adjacent vertices have comparable (by set inclusion) closed neighbourhoods then remove the (resp. one) vertex with largest neighbourhood.
Important Rules of bei

Maximum Degree Rule for Maximum Degree at least 8

If $G$ has maximum degree at least 8 then choose a vertex $w$ of maximum degree and branch to $G - N[w]$ (select $w$) and $G - w$ (discard $w$).
How to construct lower bound graphs?

Lower bound graphs of maximum degree 8 ...

... to avoid “second half of algorithm”.

Make sure that reduction rules cannot be applied.
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A Polynomial Example

LB graphs?

For all positive integers $n$, $G_n = (\{1, 2, \ldots, n\}, E_8)$, where

$$\{i, j\} \in E_8 \iff |i - j| \leq 8.$$ 

- Minimum degree 8
- Algorithm bei solves MIS on $G_n$ in polynomial time; mainly due to domination rule.
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Lower Bound Graphs

- **LB graphs**: For all positive integers $n$, $G_n = (\{1, 2, \ldots, n\}, E)$, where
  \[
  \{i, j\} \in E \iff |i - j| \in \{2, 3, 4, 5\}.
  \]

- **Tie break**: For graphs of maximum degree 8, the algorithm chooses smallest (resp. leftmost) maximum degree vertex for branching.
A search tree

```
select
17
(−11)

discard
6

select
discard
18
13
(−12)
(−7)
```
A Lower Bound

Definition
Let $T(n)$ be the number of leaves in the search tree obtained when executing algorithm bei on input graph $G_n$ using the specified tie break rules.

Recurrence

$$T(n) = T(n - 7) + T(n - 11) + T(n - 12)$$

Lower Bound of bei
The running time of algorithm bei is $\Omega(2^{0.16297n})$.
A Lower Bound

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REMINDER: Upper Bound $O(2^{0.303n})$. 
A more complicated construction seems to provide a lower bound of $\Omega(2^{0.17724n})$ for $bei$.

Find better lower bounds!
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Find better lower bounds!
Definition (Dominating Set)

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ of vertices of $G$ is a **dominating set** of $G$ if every vertex of $V - D$ is adjacent to a vertex of $D$.

Definition (Minimum Dominating Set (MDS))

Given a graph $G = (V, E)$, compute a minimum dominating set of $G$. 

![Graph Diagram]
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Given a graph $G = (V, E)$, compute a minimum dominating set of $G$. 

![Graph with vertices a, b, c, d, e, f connected in a specific pattern]
Minimum Set Cover

Definition (Set Cover)
Let \( U \) be a universe of elements and let \( S \) be a collection of (non-empty) subsets of \( U \). A subset \( S' \subseteq S \) is a set cover of \((U, S)\) if every element of \( U \) belongs to a set of \( S' \), i.e. \( \bigcup_{S \in S'} S = U \).

Definition (Minimum Set Cover (MSC))
Given a universe \( U \) and a collection \( S \) of (non-empty) subsets of \( U \), compute a minimum set cover \( S' \) of \((U, S)\).
Reducing Minimum Dominating Set to Minimum Set Cover

**Reduction of MDS to MSC**

**Minimum Dominating Set** for input graph $G = (V, E)$ reduces to **Minimum Set Cover** by setting $U = V$ and $S = \{N[v] | v \in V\}$.

Hence $D$ is a minimum dominating set of $G = (V, E)$ iff $S' = \{N[v] | v \in D\}$ is a minimum set cover of $(V, \{N[v] | v \in V\})$. 
Reducing Minimum Dominating Set to Minimum Set Cover

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**Minimum Dominating Set** for input graph $G = (V, E)$ reduces to **Minimum Set Cover** by setting $U = V$ and $S = \{N[v] \mid v \in V\}$.

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The Algorithm \texttt{msc}

1 \textbf{int} \texttt{msc}(S) \{ \\
2 \quad \textbf{if}(|S| = 0) \textbf{return} 0 ; \\
3 \quad \textbf{if}(\exists S, R \in S : S \subseteq R) \textbf{return} \texttt{msc}(S\{S}) ; \\
4 \quad \textbf{if}(\exists u \in \mathcal{U}(S) \exists \text{ a unique } S \in S : u \in S) \\
5 \qquad \textbf{return} 1+\texttt{msc}(\texttt{del}(S, S)) ; \\
6 \quad \text{take } S \in S \text{ of maximum cardinality} ; \\
7 \quad \textbf{if}(|S| = 2) \textbf{return} \texttt{poly-msc}(S) \\
8 \quad \textbf{return} \min\{\texttt{msc}(S\{S}), 1+\texttt{msc}(\texttt{del}(S, S))\} ; \\
9 \}

Reduction and Branching Rules

**Reduction Rule I**

If two sets $R, S \in S$ are comparable by set inclusion then remove the smaller one from $S$.

**Reduction Rule II**

If an element $u$ belongs to precisely one set $S \in S$ then “select $S$”. (i.e. removal of $S$ from $S$, and removal of each element of $S$ from the universe)
Reduction and Branching Rules

**Reduction Rule I**

If two sets $R, S \in \mathcal{S}$ are comparable by set inclusion then remove the smaller one from $S$.

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If an element $u$ belongs to precisely one set $S \in \mathcal{S}$ then “select $S$”. (i.e. removal of $S$ from $\mathcal{S}$, and removal of each element of $S$ from the universe)
Branching Rule (Maximum Size Rule)

Choose a set $S$ of $S$ of maximum size and branch on $S$: “select $S$”, “discard $S$”.
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Preferences of Rules are Crucial!
Analysis of Running Time (Upper Bound)

Measure & Conquer
Sophisticated running time analysis using non standard measure

Upper Bound for msc
Algorithm msc solves MINIMUM SET COVER in time $O(2^{0.305(|U| + |S|)})$.  
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Algorithm msc solves Minimum Set Cover in time $O(2^{0.305(|U|+|S|)})$. 
Analysis of Running Time (Upper Bound)

Using \text{msc}

Combine reduction from \textsc{Minimum Dominating Set} to \textsc{MSC}
with algorithm \text{msc} to obtain algorithm \text{mds}

Upper Bound for \text{mds}

Algorithm \text{mds} solves \textsc{Minimum Dominating Set} in time
\(O(2^{0.305(2^n)}) = O(2^{0.610n}).\)
Analysis of Running Time (Upper Bound)

Using $\text{msc}$

Combine reduction from Minimum Dominating Set to MSC
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Algorithm $\text{mds}$ solves Minimum Dominating Set in time $O(2^{0.305(2^n)}) = O(2^{0.610n})$. 
How to construct lower bound graphs?

Make sure that reduction rules cannot be applied.

Lower bound graphs should have small maximum degree.
How to construct lower bound graphs?

- Make sure that reduction rules cannot be applied.
- Lower bound graphs should have small maximum degree.
Lower Bound Graphs

- $a_k$ to $b_k$
- $a_2$ to $b_2$
- $a_1$ to $b_1$
- $c_1$ to $b_1$
- $c_2$ to $a_1$
- $c_k$ to $c_2$
Important Properties of Lower Bound Graphs

1. $G_k$ has maximum degree 4.
2. Each set $S_v := N[v]$ of $S$ has size at most 5.
3. There is a (bad) execution of msc (resp. mds) on $G_k$ choosing the branching vertices such that no reduction rule (not even a connectedness rule) can ever be applied.
How to Choose a Selection Rule?

- A selection rule specifies how ties will be broken in a (bad) execution of the algorithm.

**Objective I**

We (i.e. an adversary of the algorithm) choose a selection rule of the algorithm, specifying how to break ties, such as to maximize the running time of the algorithm.

**Objective II**

The number of leaves in the search tree obtained when mds on $G_k$ uses the specified selection rule should be as large as possible.
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Objective II
The number of leaves in the search tree obtained when mds on $G_k$ uses the specified selection rule should be as large as possible.
Our Selection Rule for mds

Round $i$, $i \in \{2, 3, \ldots, n - 1\}$: treat a pair $P = \{x_i, y_i\}$ of vertices belonging to triangle $T_i = \{a_i, b_i, c_i\}$.
Initially $P = \{a_2, b_2\}$.

For each pair $P = \{x_i, y_i\}$ branch in the following 3 ways:

1) select $S_{x_i}$
2) discard $S_{x_i}$, and then select $S_{y_i}$
3) discard $S_{x_i}$, and then discard $S_{y_i}$

The following new pairs of vertices correspond to each of the three branches:

1) $P_1 = \{a_{i+2}, b_{i+2}, c_{i+2}\} \setminus x_{i+2}$
2) $P_2 = \{a_{i+2}, b_{i+2}, c_{i+2}\} \setminus y_{i+2}$
3) $P_3 = \{x_{i+1}, y_{i+1}\}$

On each pair $P_j$ recursively repeat the process.
Definition

Let $T(k)$ be the number of leaves in the search tree obtained when algorithm mds on input graph $G_k$ uses the specified selection rule.

Of the three branches of $T_i$ two are proceeded on $T_{i+2}$ and one is proceeded on $T_{i+1}$.

Recurrence

$T(k) = 2 \cdot T(k - 2) + T(k - 1)$
A Lower Bound

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Lower Bound of \textit{mds}

The running time of algorithm \textit{mds} is $\Omega(2^k) = \Omega(2^{n/3})$. 
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Lower Bound of $\text{mds}$

The running time of algorithm $\text{mds}$ is $\Omega(2^k) = \Omega(2^{n/3})$.

REMINDER: Upper Bound $O(2^{0.610n})$. 
Hopefully ... 

the insights obtained when studying lower bounds for a particular Branch & Reduce algorithm might be of help for choosing sophisticated measures to improve the upper bound of the (worst case) running time.
Conclusions

Study and Improve Lower Bounds!

Provide lower bounds with exponential-time algorithms!
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Conclusions

What we really need?

We do need better methods to analyse the running time of Branch & Reduce algorithms.
Thank you for your attention!
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For Further Reading II

Kratsch, D., and M. Liedloff,
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