About the domino problem in the hyperbolic plane, a new solution: complement

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May 8, 2007

Abstract

In this paper, we complete the construction of paper [9, 11]. Together with the proof contained in [9, 11], this paper definitely proves that the general problem of tiling the hyperbolic plane with *à la* Wang tiles is undecidable.

1   Introduction

The question, whether it is possible to tile the plane with copies of a fixed set of tiles was raised by Wang, [14] in the late 50's of the previous century. Wang solved the *partial* problem which consists in fixing an initial finite set of tiles: indeed, fixing one tile is enough to entail the undecidability of the problem. The general case, later called the **general tiling problem** in this paper, GTP in short, without condition, in particular with no fixed initial tile, was proved undecidable by Berger in 1966, [1]. Both Wang's and Berger's proofs deal with the problem in the Euclidean plane. In 1971, Robinson found an alternative, simpler proof of the undecidability of the general problem in the Euclidean plane, see [12]. In this 1971 paper, he raises the question of the general problem for the hyperbolic plane. Seven years later, in 1978, he proved that in the hyperbolic plane, the partial

problem is undecidable, see [13]. Up to now, and as far as I know, GTP remained open.

In this paper, we complete the proof that GTP is also undecidable in the case of the hyperbolic plane which is given in [9, 11].

In a first section, we sketchilly remember the construction of [9, 11] and we very briefly remember the reader the construction of the mantilla and its properties already proved in [8, 10].

In the second section, we give the needed complement. This will completely prove that:

Theorem 1 *The general problem of tiling the hyperbolic plane is undecidable.*

Then, we conclude with remarks on further improvements and a few corollaries which we already obtained from the theorem.

In this section, first, we very briefly mention the construction of the mantilla, the basic frame in which the different implementations performed by our construction take place.

In the next sub-section, we briefly remind the **abstract brackets** which is the key new tool of the general frame of the proof. This one-dimensional construction is mentioned in [5], and it is at the basis of Berger's proof of GTP for the Euclidean plane. Robinson's proof of GTP for the Euclidean plane is based on a two-dimensional adaptation of the one-dimensional argument. Paper [5] focuses at the two-dimensional construction and gives a deep account on this situation, especially from an algebraic point of view.

Then , we briefly look at implementation of the one-dimensional construction in the Euclidean plane, lifting the intervals of this model into triangles. Such a construction is called an **infinite model**. This implementation is transported into the hyperbolic plane, infinitely many times. This entails a kind of cutting of the construction which we analysed in our study of the one-dimensional construction under the name of **semi-infinite model**. The final step of the construction consists in indicating a way to **synchronize** all these implementations in such a way that they appear as different cuts of a single infinite model. The key property of these triangles is that there are infinitely many of them for infinitely many heights.

The last point is to implement a grid in each of these domains. It allows to implement a space time diagram of the same Turing machine, as in the classical proofs of Berger and Robinson. The complement does not change this part of the proof which remains what it is in [9, 11].

The reader is invited to look at the technical report, [11] on which the paper is based and which is available at the following address:

where full proofs can be found of what is indicated in this section.

1.1 The mantilla

Here, we consider the tessellation $\{7, 3\}$ of the hyperbolic plane, which we call the **ternary heptagrid**, simply **heptagrid**, for short, see [2, 7]. It is generated by the regular heptagon with vertex angle $\dfrac{2\pi}{3}$ by reflections in its sides and, recursively, of the images in their sides.

1.1.1 The flowers

In the ternary heptagrid, a **ball** of **radius** n around a tile T_0 is the set of tiles which are within distance n from T_0 which we call the **centre** of the ball. The **distance** of a tile T_0 to another T_1 is the number of tiles constituting the shortest path of adjacent tiles between T_0 and T_1. We call **flower** a ball of radius 1.
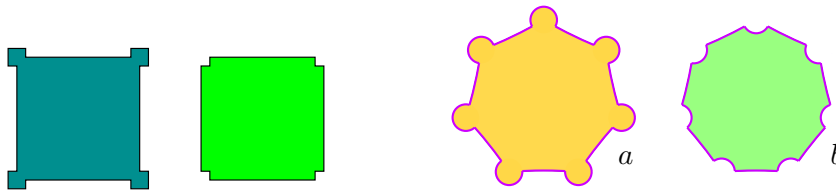

Figure 1 *On the left: Robinson's basic tiles for the undecidability of the tiling problem in the Euclidean case. On the right: the tiles a and b are a 'literal' translation of Robinson's basic tiles to the situation of the ternary heptagrid.*
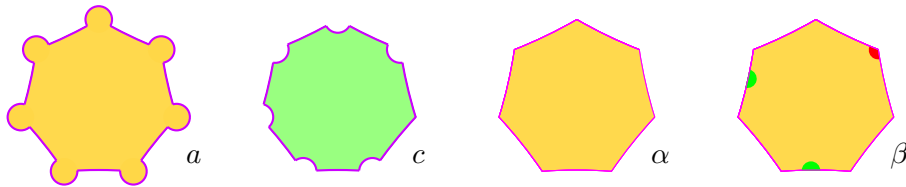

Figure 2 *On the left: change in the tiles à la Robinson. On the right: their translation in pure Wang tiles.*

The mantilla consists in merging flowers in a particular way. It comes from an attempt to implement Robinson's construction in the Euclidean

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plane based on the left-hand side tiles of figure 1. The right-hand side tiles of the figure are their 'literal' translation. It is not difficult to see that it is not possible to tile the hyperbolic plane with tiles a and b. However, a slight modification of the tile b, see the tile c in figure 2, leads to the solution.

On the right hand side of figure 2, we have the transformation of tiles a and c into Wang tiles. We call the tile α a **centre** and the tile β a **petal**. We refer the reader to [10, 11] for the numbering technique allowing to force the tiles β to be put around tiles α. Now, a petal belongs to three flowers at the same time by the very definition of the implementation. From this, there is a partial merging of the flowers.

It is not difficult to see that there can be several types of flowers, considering the number of red vertices for which the other end of an edge is a vertex of a centre. We refer the reader to [10] for the corresponding properties. Here, we simply take into consideration that we have three basic patterns of flowers, which we call F-, G and **8**-flowers respectively. They are represented by figure 3.

The figure also represents the way which allows to algorithmically construct the tiling resulting from the tiles α and β which we call the **mantilla**. It consists in splitting the **sectors** generated by each kind of flowers in subsectors of the same kind and only them, which we call the **sons** of the flower. From this, we easily devise a way to recursively define a tiling. The construction is deterministic below the flower, and it is non-deterministic when we proceed upwards. We do not make the notion of top and bottom more precise: it will be done later. The exact description of the splitting can be found in [10]. We simply remark that such a splitting is an application of the general method described in [6], for instance.
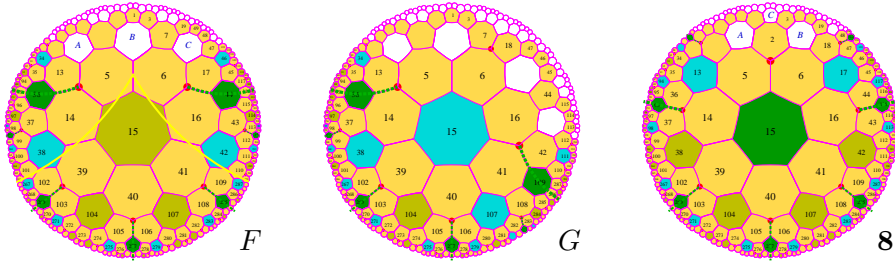

Figure 3 *Splitting of the sectors defined by the flowers. From left to right: an F-sector, G-sector and **8**-sector.*

Based on these considerations, we have the following result which is thoroughly proved in [10]:

Lemma 1 *There is a set of* 4 *tiles of type* α *and* 17 *tiles of type* β *which allows to tile the hyperbolic plane as a mantilla. Moreover, there is an algorithm to perform such a construction.*

1.1.2 Trees of the mantilla

Note that the left-most flower of figure 3, which represents an F-sector, also indicates a region delimited by continuous lines, yellow in coloured figures. This lines are **mid-point** lines, which pass through mid-points of consecutive edges of heptagons of the heptagrid. As shown in [2, 7], they delimit a Fibonacci tree. The tiles inside the tree which are cut by these mid-point rays are called the **borders** of the tree, while the set of tiles spanned by the Fibonacci tree is called the **area** of the tree.

Say that an F-son of a G-flower is a **seed** and the tree, rooted at a seed is called a **tree of the mantilla**. As the seeds are the candidates for the construction of a computing region, they play an important rôle. From figure 3 we can easily define the **border** of a sector which is a ray crossing **8**-centres. See [10] for exact definitions.

Lemma 2 *The borders of a tree of the mantilla never meet the border of a sector.*

From lemma 2, as shown in [10], we easily obtain:

Lemma 3 *Consider two trees of the mantilla. Their borders never meet. Either their areas are disjoint or the area of one contains the area of the other.*

From this, we can order the trees of the mantilla by inclusion of their areas. It is clear that it is only a partial order. We are interested by the maximal elements of this order. We call them **threads**, see [10] for an exact definition. Threads are indexed by $I\!N$. Among them, there can be a unique **ultra-thread** which is indexed by $\mathbb{Z}$. Note that the union of the areas of the trees which belong to an ultra-thread is the hyperbolic plane. There can be realizations of the mantilla with or without an ultra-thread.

1.1.3 Isoclines

In [11], we have a new ingredient. We define the status of a tile as **black** or **white**, defining them by the usual rules of such nodes in a Fibonacci tree. Then, we have the following property:

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Lemma 4 *If a seed is a black tile, all other seeds in the area of the tree of the mantilla which it delimits are black tiles. Also, within the same area, the **8**-centres are all black tiles.*
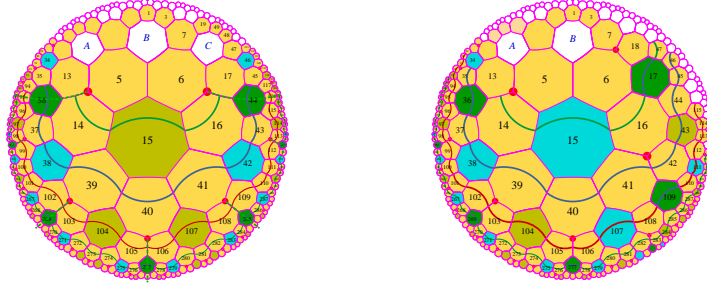

Figure 4 *The black tile property and the levels:*
*On the left-hand side, a black F-centre; on the right-hand side, a black G_ℓ-centre. We can see the case of an **8**-centre on both figures.*

As shown in [11], we can define arcs as follows: in a white tile, the arc joins the mid-points of the sides which have a common vertex with the side shared by the father. In a black tile, the arc joins the mid-point of the sides which are separated by the side shared by the father and the side shared by the uncle, which is on the left-hand side of the father. Joining the arcs, we get paths. The maximal paths are called **isoclines**. They are illustrated in figure 4. An isocline is infinite and it splits the hyperbolic plane into two infinite parts. The isoclines from the different trees match, even when the areas are disjoint.

Lemma 5 *Let the root of a tree of the mantilla T be on the isocline 0. Then, there is a seed in the area of T on the isocline 5. If an **8**-centre A is on the isocline 0, starting from the isocline 4, there are seeds on all the levels. From the isocline 10 there are seeds at a distance at most 20 from A.*

We number the isocline from 0 to 19 and repeat this, periodically. This allows to give sense to **upwards** and **downwards** in the hyperbolic plane.

1.2   The abstract brackets

We refer the reader to [11] for an exact definition. However, figure 5, below, illustrates the construction which now, we sketchily describe.

The generation 0 consists of points on a line which are regularly spaced. The points are labelled R, M, B, M, in this order, and the labelling is

periodically repeated. An interval defined by an R and the next B, on its right-hand side, is called **active** and an interval defined by a B and the next R on its right-hand side is called **silent**. The generation 0 is said to be **blue**.
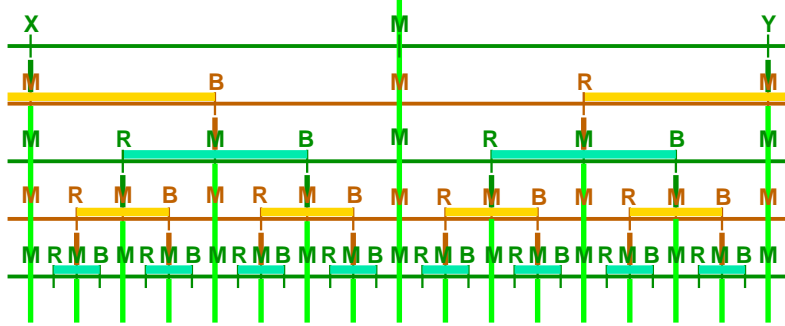

Figure 5 *The silent and active intervals with respect to mid-point lines. The light green vertical signals send the mid-point of the concerned interval to the next generation. The colours are chosen to be easily replaced by red or blue in an opposite way. The ends X and Y indicate that the figure can be used to study both active and silent intervals.*

Blue and red are said **opposite**. Assume that the generation n is defined. For the generation $n+1$, the points which we take into consideration are the points which are still labelled M when the generation n is completed. Then, we take at random an M which is the mid-point of an active interval of the generation n, and we label it, either R or B. Next, we define the active and silent intervals in the same way as for the generation 0. The active and silent intervals of the generation $n+1$ have a colour, opposite to that of the generation n.

When the process is achieved, we get an **infinite model**. The model has interesting properties, see [11]. We cannot mention all of them here. We postpone some of them to the Euclidean implementation with triangles.

In an interval of the generation n, consider that a letter of a generation m, $m \leq n$, which is inside an active interval is hidden for the generations k, $k \geq n+1$. Also, a letter has the colour of its generation. Now, we can prove that in the blue active intervals, we can see only one red letter, which is the mid-point of the interval. However, in a red active interval of the generation $2n+1$, we can see $2^{n+1}+1$ blue letters.

Cut an infinite model at some letter and remove all active intervals which contain this letter. What remains on the right-hand side of the letter is called a **semi-infinite model**.

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It can be proved that in a semi-infinite model, any letter y is contained in at most finitely many active intervals, see [11].

1.3   Interwoven triangles

Now, we lift up the active intervals as **triangles** in the Euclidean plane. The triangles are isoceles and their heights are supported by the same line, called the **axis**, see figure 6.

We also lift up silent intervals of the infinite model up to again isoceles triangles with their heights on the axis. To distinguish them from the others, we call them **phantoms**. We shall speak of **trilaterals** for properties shared by both triangles and phantoms.
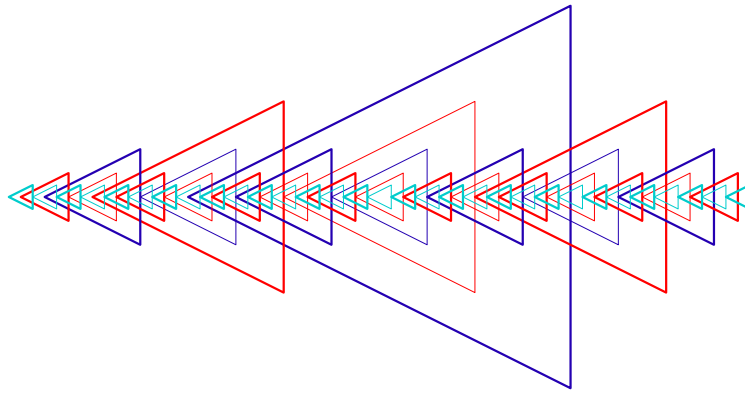

Figure 6 *An illustration for the interwoven triangles.*

We have very interesting properties for our purpose.

Lemma 6 *Triangles of the same colour do not meet nor overlap: they are disjoint or embedded. Phantoms can be split into **towers** of embedded phantoms with the same mid-point and alternating colours. Trilaterals can meet by a basis cutting the half of leg which contains the vertex.*

From these properties, we prove in [11] that:

Lemma 7 *the Euclidean plane which can be forced by set of* 190 *tiles.*

In [11], we display the corresponding tiles which are in a square format, and we also describe them with the help of formulas taking into account the properties of lemma 6.

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1.4 Hyperbolic implementation and the computing areas

We implement the interwoven triangles in the hyperbolic plane by using the trees of the mantilla as frames for the legs of the trilaterals. The basis is materialized by the trace of an isocline in the area of the trilateral.

1.4.1 The synchronization

The axis will be somehow materialized by a thread. As most threads are indexed by $I\!N$ only, we have always the implementation of a semi-infinite model. Now, we shall manage the implementation in such a way that the semi-infinite models are simply different cuts of the same infinite model. The possibility of the realization of the infinite model in the case of an ultra-thread brings in no harm.

To achieve this point, we very briefly indicate a feature of the tiles. The legs of a triangle emit horizontal signals **outside** the triangle. The signals have the same colour as the emitting triangle and they have a laterality. The left-hand side leg emits left-hand side signals, the right-hand side leg emits right-hand side ones. Both kinds of signal cross the tiles in an **upper** or **lower** position, always at the lower one for the vertex. Phantoms also emit signals, only at the vertex, in a **lower** position, and at the corner of the basis, in an **upper** position.

The tiling forces the construction of trilaterals generation after generation. A vertex of the next generation grows legs downwards until they meet the green signal which indicates the mid-point of the legs. Triangles stop their green signal, phantoms do not.

To synchronize the semi-infinite models, bases of triangles which are on the same isocline merge. The distinction between outside and inside a triangle is given by the presence or absence of the upper horizontal signal of the same colour as the basis. We say that the basis is **covered** or **open**. Inside a triangle, the left-hand side and right-hand side signals can be joined only at a vertex, and so, they must be lower signals. Outside a triangle, horizontal signals of different lateralities can be joined, as the directions from where the signals come are the opposite with respect to what happens inside a triangle. The needed tiles are provided only for meetings outside trilaterals.

Now, the distinction between a covered and an open basis allows the implementation of the construction using the tiles devised for lemma 7, using the same algorithm of construction. Indeed, first halves of legs, *i.e.* from the vertex to the mid-point, may cut bases, either covered or open,

leaving them covered or open respectively. The change to the second half is triggered by the detection of the green signal. Next, the second half meets covered bases. The first open basis, necessarily of its colour, is the expected basis for this trilateral.

Note that inside a trilateral and between the same set of isoclines, there are several triangles of former generations. In the next section we manage this point for which the complement given in this paper is needed.

1.4.2 The computing areas

They are defined by the **active** seeds which we now define.

By definition, we decide that all seeds which are on an isocline 0 are active. This is enough to guarantee that the set of active seeds is dense in the hyperbolic plane. Next, an active seed diffuses a **scent** inside its trilateral until the isocline 5, starting from this seed, is reached. Seeds which receive the scent, and only them, become active. An active seed also triggers the green signal. By construction, The generation 0 is not determined by the meeting of a green signal. But the others are.

We can see that the scent process constructs a tree. The branches of the tree materialize the thread which implements the considered semi-infinite model. Note that the above synchronization mechanism fixes things for spaces between triangles but also inside them.

An important mechanism provided by the tiles of lemma 7 is the detection of the **free rows** inside **red** triangles. These free rows are the isocline whose projection on the axis is a blue letter, visible in the active interval defined by the height of the triangle. It is not difficult to provide tiles for that, also based on the red horizontal signals of different lateralities and positions, see [11].
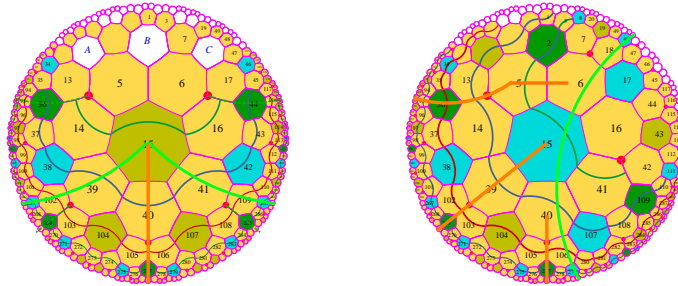

Figure 7 *The perpendicular starting from a point of the border of a triangle which represents a square of the Turing tape.*
On the left-hand side: the case of the vertex. On the right-hand side, the three other cases for the right-hand side border are displayed on the same figure.

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The free rows inside the red triangles are the horizontal of our computing areas. It remains us to define the verticals which are needed for implementing the space-time diagram.

The verticals consist into rays which cross **8**-centres. Figure 7 illustrates how they are connected to the different possible cases of contact of the isocline of a free row with the border of the tree.

The computing signal starts from the seed. It travels on the free rows. Each time a vertical is met, which contains a symbol of the tape, the required instruction is performed. If the direction is not changed and the corresponding border is not met, the signal goes on on the same row. Otherwise, it goes down along the vertical until it meets the next free row. There, it looks at the expected vertical, going in the appropriate direction. Further details are dealt with in [11] and are rather close to the classical proofs.

2 The complement

In this section, we deal with the tuning promised in sub-section 1.4.1, about the description of the synchronization mechanism.

2.1 The point to be tuned

In fact, in sub-section 1.4.1, we describe the synchronization problems raised by the bases and vertices, in order to obtain that all the threads implement a cut of the same infinite model of the abstract brackets. Of course, if the bases and vertices are synchronized, the mid-lines of the trilaterals are also synchronized. As mentioned in sub-section 1.4.1, this time we have infinitely many copies of the same trilateral within a same set of isoclines: call such a set a **latitude**. Now, we have to closer look at the possible interactions between the trilaterals which occur within a fixed latitude. Also call **amplitude** the number of isoclines contained in a given latitude. The problem is that in between two contiguous triangles of the same latitude Λ, there may be and, usually, there are trilaterals of smaller generations, whose latitude is contained in Λ. Now, a part of the phantoms within Λ have the same mid-line as the triangles whose height is the amplitude of Λ. Now, when we consider the phantoms which are crossed by this mid-line between two consecutive triangles T_1 and T_2, the green line which they emit runs along the mid-line. As the legs of a phantom do not stop the green signal, nothing prevent them to meet the legs of T_1 and T_2. Now, they should not meet these legs as the legs of a triangle stop the green signal.

2.2   A possible mechanism

The mechanism is the following. Any triangle T of a given latitude Λ whose height is the amplitude of Λ, stops the green signal which runs on its mid-line. Now, outside its legs, on the mid-line, T stretches out two antennas: a left-hand- and a right-hand side one. They rôle is to detect, the anteanna sent by the next triangle of the same latitude in the direction followed by the antenna.

What we need is a characterization of the structure of the trilaterals within a given latitude. This is provided by the following lemma:

Lemma 8 *Let T and T' be two consecutive triangles of the same generation n and within the same latitude Λ whose amplitude is the common height of T and T'. Let A be the mid-point of the right-hand side leg of T and let B be the mid-point of the left-hand side leg of T'. Then, there is a tile C and a tile D on the isocline passing through A and B such that the interval $[C, D]$ is contained in the interval $[A, B]$ and: if a leg of a trilateral of a generation m, with $m > n$, crosses $]A, C[$, $]D, B[$ respectively, then it is a right-hand side leg, a left-hand side leg respectively, of this trilateral. Moreover, $[C, D]$ is not crossed by any leg of a trilateral belonging to a generation m, with $m > n$. A leg ending with a corner within $[A, C]$ or $[D, B]$ is considered as a crossing leg.*

Proof. Let I be the isocline which contains A and B. Between A and B, I crosses several legs of trilaterals. It crosses both legs of a trilateral if and only if the trilateral belongs to a smaller generation: it is contained in Λ. In this case, the trilateral is a phantom. It cannot be a triangle: smaller triangles have their projection within the projection of a half leg of T and so, they cannot meet I.
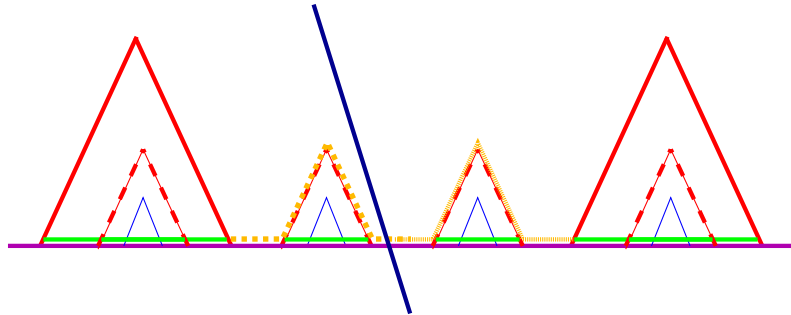

Figure 8 *The principle of the antennas.*

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Now, a triangle of the next generation would be raised in the mid-line of a triangle K of Λ and inside K. This is impossible as there is no triangle of Λ between A and B. And for still bigger trilaterals, if both legs are crossed, the trilateral contains a triangle of the generation of T within Λ too. Indeed, once a trilateral exists, it contains all possible trilaterals which can be contained within its area.

Accordingly, when I crosses both legs of a trilateral, it is a phantom and, more over, the mid-line of this phantom is supported by I. This is a consequence of the properties of silent intervals.

From this analysis, we conclude that there is a tile C on I between A and B such that on the left-hand side of C with the following property. Considering the crossing of I, if any, by legs of trilaterals of bigger generation, we have that the intersected right-hand side legs are all on the left-hand side of C and that the intersected left-hand side legs are all on the right-hand side of C. In fact there are several such tiles C which constitute an interval of I, between A and B. Let us denote this interval by $[C, D]$. ∎

As phantoms sharing the same mid-line are constituted in towers, within a latitude Λ, a tower is necessarily finite and so, it contains an **eldest** phantom.

The antenna coming from T will cross the right-hand side legs which are on the left-hand side of C and it will jump over all eldest phantoms whose both legs are crossed by I and which stand between A and C. The antenna coming from T' will do the same for the eldest phantoms whose both legs are crossed by I and which stand between B and D. In between C and D, there are only phantoms within Λ. The antennas go on jumping other the eldest ones they meet until both antennas meet on some tile of $[C, D]$, outside any phantom, see figure 8.

Remark. We can prove that in between two contiguous triangles of the same generation and within the same latitude Λ, there is at most one leg of a higher generation crossing A and B: either of the generation $n+1$ or of the generation $n+2$. It belongs to a generation generated by one of the two triangles or generated by the former one.

This can be proved as follows.

First, say that a **triangle** T is the **father** of a **trilateral** K if T is of the generation n and K is of the generation $n+1$ and if the vertex of K is on the mid-line of T inside T. Note that a trilateral has a unique father and that a triangle has a lot of sons. We can repeatedly apply this definition to a trilateral K, leading to a sequence $T_0, \ldots T_m$ of triangles with T_0 of the generation 0, such that T_i is the father of T_{i+1}, for $i \in \{0..m-1\}$ and that $T_m = K$. In this case, T_0 is called the **remotest ancestor** of K.

From the study performed in [11] and from the results indicated in

sub-ection 1.2 of this paper, the height of a trilateral of the generation n is $2^{n+1}+1$ if measured by the number of the isoclines crossing its legs, vertex and basis being included in this account. Now, it is not difficult to show that the distance of the mid-line of a trilateral to the vertex of its remotest ancestor is 2^{n+1}, measured in the same way: we count the isoclines crossing the legs from the mid-line to the vertex of the remotest ancestors, the last vertex and the initial mid-line being taken into account.

Let T_1 and T_2 be the contiguous triangles of exactly the same latitude and let A and B as in lemma 8. Assume that a trilateral P whose basis is also generated by T_1 exists and that the right-hand side leg δ of P crosses AB, taking into account that, at a corner on AB, we consider that the leg crosses AB. Then, the ancestor A_1 of T_1 is on the closest isocline 0 to the mid-line λ of P, under λ, and the ancestor A_2 is also on this isocline. Now, A_2 is outside P, otherwise T_2 would also be inside P, which contradicts the assumption. Now, there is no other seed on the isocline 0 between the vertices of A_1 and A_2. If there would be another one σ, σ would be the vertex of a triangle A_3 of the generation 0 and, the same trilaterals as those occuring inside the trees rooted at the vertices of A_1 and A_2 would also occur inside the tree rooted at the vertex of A_3. In particular, there would be a triangle T_3 of the generation of T_1 and in the same latitude and which would stand between T_1 and T_2. This is impossible by our assumption.

Now, if there would be another leg ℓ crossing AB belonging to a trilateral of the generation m with $m > n$, then ℓ would be between δ and A_1, for instance. It is easy to see that it must be a right-hand side leg. Otherwhise, the trilateral Q defined by ℓ would contain A_2 as, were it not be the case, this trilateral would contain a copy of T_1 which would stand in between T_1 and T_2, again a contradiction. But if Q contains A_2, it also contains a trilateral generated by T_2, which is a copy R of P and, necessarily, R would stand outside P. Now, whatever the distance between the roots of P and Q, the distance at the mid-points, which are on the same isocline is much bigger, and there is room for an active seed A_3 in between A_1 and A_2: see, below, the table of distances between the border of a tree and the closest outside seed on an isocline 0: 2, 36 or 269. This would again contradict our assumption on T_1 and T_2.

And so, we may assume that ℓ is the right-hand side leg of a trilateral of the generation $n+2$ generated by the mid-line of P and inside P. Is this possible?

The mid-line of P is λ. Let ν be its closest active seed near the right-hand side leg ℓ of P. The problem which we have to consider is the distance between the closest seed to ℓ. We measure this distance in the number of nodes from ℓ to the seed which are on the same isocline. In fact, we have to consider the distance on λ and on the next isocline 0.

From the study of [11] performed with the help of a computer programme, and taking into account the periodicity of the tiles on the border of a tree of the mantilla, the considered distances are given by the following table:

(λ), 15:	2	36	269
0:	36	269	2

Also note that for the isoclines 15, these distances give the closest seed, which is not necessarily active. Accordingly, we can see that, whatever the distance of the closest active seed σ to the leg ℓ of the generation $n+1$ on the isocline 15, the closest seed to ℓ on the isocline 0 is in between the right-hand side leg of the tree rooted at σ and ℓ. This means that the remotest ancestor A_1 of T_1 is not inside the tree rooted at σ. And so, the righ-hand side border of this tree is on the left-hand side of the left-hand side border of T_1.

This indicates that another possible leg of a higher generation between T_1 and T_2 could be of the generation $n+2$ and, precisely, in such a case, the remotest ancestor of T_1 would be the rightmost side on the isocline 0 inside the tree rooted at σ.

Now, the distances of the closest seeds on the isocline 15 and the next isocline 0 outisde a leg of a trilateral are given by the same table as above. Accordingly, between δ and T_1, there cannot be a leg of the generation $n+2$. Higher generations are a fortiori ruled out: otherwise, the generation $n+2$ would also be present.

A last point to notice is that there cannot be two legs of the generation $n+1$: a right-hand side one on the side of T_1 and a left-hand side one on the side of T_2.

Now, the distance between legs of such opposite legs of the same generation is increasing as we go down along the isoclines. The closest distance is the smallest distance bewteen two active isoclines. It is not difficult to see that it is realized by the F-sons of a G_r- and a G_ℓ-centres which belong to the same F-flower when the F-centre is a black node. The distance is then 26. At the level of λ, the distance is much bigger than the biggest distance 269 indicated by the above table.

Accordingly, two such legs cannot be present between T_1 and T_2.

And so, our claim is proved.

Next, we provide tiles to implement this mechanism and then, in the next subsection, we check that the new mechanism does not disturb the general construction, outlined in the previous section.

2.3 The tiles

The antenna is given a specific colour, we call it **orange**, and it has a laterality: there is an right-hand side antenna, which goes to the right and a left-hand side one, which goes to the left. Due to the colour of the antenna, we shall often speak of the **orange signal** and of its laterality.

The first principle is that an antenna cannot directly be in contact with the green signal. And so the green signal and the orange one are always separated by the leg of a trilateral, more precisely, they both occur at the mid-point of a leg. The other principle is that the antenna is stopped, at

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one end by a mid-point of a triangle and, at the opposite end, by an antenna of its opposite laterality.
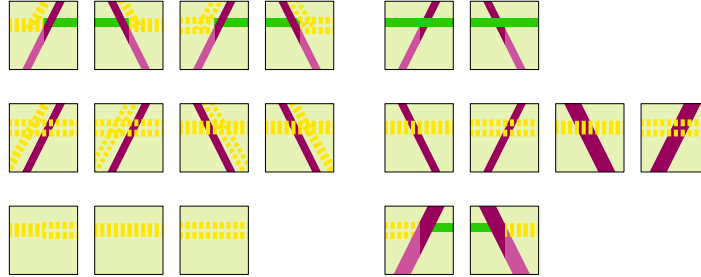
the signals for the antennas:

Figure 9 *The tiles for the antennas:*
Number them by rows and columns, in 1..3 *for rows and* 1..6 *for columns.*
In the row 3, *we have the ends of the antennas: the join tile,* 1−1 *and the mid-point of a triangle,* 3−5 *and* 3−6.
In the row 1, *we have the tile to jump over an eldest phantom.*
In the row 3, *we have the tiles to cross legs of bigger trilaterals. Note that not any combination of lateralities for the leg and for the antenna are permitted. Note that the two tiles for the vertices are not represented, as this is easy.*

Now, if the antenna meets a leg at a mid-point, and there is necessarily a green signal on the other side of the leg, then the orange signal climbs up along the leg.

The tiles are dispatched by figure 9.

2.4 Checking the correctness

We have now to check that the set of tiles given by figure 9 force the above specifications given to the antennas. Note that the tiles are in fact **meta-tiles**. We represented the colour by a variable colour. Note also that in the tiles of the row 1, we have mid-points of phantoms, necessarily. However, in the row 2, tiles 2-5 and 2-6 belong to legs of phantoms, while tiles 2-7 and 2-8 belong to legs of triangles. In both cases, it may be either the first half or the second half of a leg. The tiles 2-1, 2-2, 2-3 and 2-4 concern phantoms only and, as the other tiles of the row, of any colour and for any half-leg. Also, the two meta-tiles for the vertices are not represented. One is endowed with the signal to the right, the other with the signal to the right. Note that a vertex cannot join antennas of different lateralities. Note that the vertex of a phantom with no orange signal over it is also available.

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It is not difficult to check that the antennas can be constructed by the considered tiles. We shall focus on the converse: nothing else can be obtained from the tiles.

First, we check that inside an eldest phantom, the phantoms are crossed at their mid-line by the green signal only. This is obtained by the combination of lateralities and the fact that both first halves of the legs of a phantom are covered by a signal with the same laterality. Also note that there is a single join-tile for horizontal parts of antennas of opposite laterality. This join-tile prevents to change of laterality inside a phantom. Accordingly, if an orange signal would cover a non eldest phantom, there would be a contradiction at one mid-point of a the phantom of the next generation: a contradiction on the left-hand side mid-point with an orange signal to the right and on the right-hand side mid-point with an orange signal to the left.

The same argument explains that the present construction does not prevent the mechanism of the green signal to detect the mid-line of a triangle. Indeed, if instead of the tile 3-5 a tile 2-8 is used, as the join tile 3-1 cannot be used inside the triangle, the orange signal, after jumping other the eldest phantoms inside the triangle could not match the meeting with the other leg of the triangle. We have a symmetrical argument if the tile 2-7 is used in place of the tile 3-6. And so, the unique solution is to use the tiles 3-5 and 3-6.

Also, as the orange signal cannot directly meet with a green one, on a given mid-line, we have either the green signal or an orange one. We have a green signal inside a triangle and inside the phantoms whose mid-line is the considered isocline.

We also note that the tiles of the row 2 cannot be used in place of those of the row 1 and that the tiles of the row 2 must be used with legs of trilaterals of a bigger generation than that of the triangles emitting the crossing signal. Note that the tiles 2-1, 2-2, 2-3 and 2-4 are used by triangles contained in an eldest phantom P. As the mid-lines of these triangles is not the mid-line of P, the signal crosses the leg which bears the orange signal covering P. Note that the opposite legs of P are crossed by signals of the same laterality as the leg and that due to the unique join tile, there must be a triangle inside the considered phantom. Also note that for the phantoms of the generation 0 this brings in no contradiction as they do not contain triangles.

At last, the start of a jump at a mid-point cannot be confused by a crossing. For instance, if the antenna to the righ from T goes to far and meets a left-hand side leg on the right-hand side of the point D defined at sub-section 2.2, then there is a trilateral which receives a green signal which

will meet the orange signal of the left-hand side antenna from T', which will produce a contradiction. and so, the single solution is to use the joining tile at a place of $[C, D]$ which is outside any phantom of the considered latitude.

And so, this proves that the antenna mechanism is forced by the set of tiles of figure 9.

Note that in the case of the butterfly model, see [9, 11], the mechanism of the antenna forces the green signal to run over the whole isocline which is the mid-point of the latitude which contains no triangle. Indeed, the laterality constraints of the tiles of the second row prevent an orange signal to run at infinity.

With this, we completed the proof of theorem 1.

Conclusion

The first consequence is that we need a bigger number of tiles than what is announced in [9, 11]. Indeed, the orange signals entails an increase of the number of vertices, of mid-points, of corners, of crossings with various legs. Signals of bases are also changed by the completion of the construction. Note that corners behave as crossing legs. Also note that the signal particularly addresses isoclines 5 and mainly 15. A detailed counting will be given in a forthcoming paper making the synthesis of [9, 11] and the present paper. However, a rough estimate shows that the number of prototiles should be now around 21,000 tiles. But, the number of meta-tiles, the variable tiles indicating the computation signs which depend on the simulated Turing machine, is not changed by the orange signal.

It is interesting to notice the rôle played by the laterality in the whole proof of theorem 1. The laterality is not used exactly in the same way in the antenna mechanism and in the mechanism of detecting the green line, the bases and the free rows. However, the same difference is used together with the possibility to connect opposite lateralities in a single way. May be a closer analysis of this mechanism could be used to reduce the number of signals, hence to reduce the number of tiles.

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