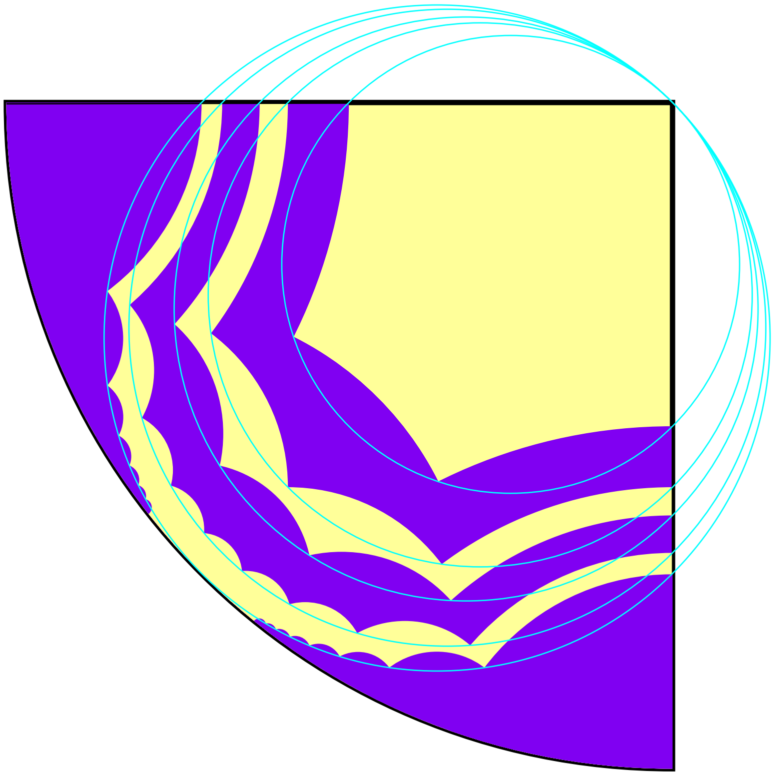

Infinigons and infinigrids in the hyperbolic plane

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Infinigons and infinigrids in the hyperbolic plane,

A combinatorial approach

Maurice MARGENSTERN

Abstract – In this technical report, we define the *infinigons*, which are a closed infinite polygonal line, that live in the hyperbolic plane. We characterise them in various ways. We also study which ones are able to generate a tiling via an iterative process of reflection in their sides and, recursively, the same starting from the images that are obtained in this way. We call such tilings *infinigrids*. We show that a spanning tree can be associated to such tilings and we give an algorithm to construct them.

Hyperbolic geometry is becoming more and more attractive, probably due to the irresistible aesthetic impression given by tilings which are obtained in the hyperbolic plane. On that regard, hyperbolic geometry of the plane is much richer than its euclidean relative. However, the latter gets its revenge in high dimensions: there are always tilings of the euclidean space of dimension p for any $p \geq 2$ which are based on a regular polytope, while for $p \geq 5$ this is never true for the hyperbolic space of dimension p, see [10].

We refer the reader to [8] and [9] for introductory material on hyperbolic geometry. In order to fix things, we shall use Poincaré's models of the hyperbolic plane. One model is in the open unit disk U of the euclidean plane. The other model is the conformal image of the previous one onto an open half-plane, the upper half-plane, for instance. Later, we shall denote the half-plane by P. The border of the disk, ∂U is called the set of *points at infinity*. Similarly, for P, the set of points at infinity is ∂P - the border of P - to which we add a point P^∞ at infinity, for compactification. In the disk model, lines are the traces in U of its diameters or the traces in U of circles that are orthogonal to ∂U. In the half-plane model, lines are verticals or half-circles whose centre lie on the line of points at infinity ∂P. These models have a very remarkable property: hyperbolic angles between lines are the euclidean angles between the corresponding circles. These models are easily generalised to higher dimensions.

As both models are equivalent under analytic mappings, we can easily go from one to another. We shall denote the hyperbolic plane by $I\!\!H^2$ and, depending on the situation, we shall consider its realizations in U or P. In the argument, we shall refer to e-lines, h-lines, e-circles or h-circles whenever appropriate, where e indicates that the object is taken in its euclidean meaning and h that it is taken in the hyperbolic sense.

As for intuition, each model has its own qualities. To see this, recall that a *horocycle* is a curve which is represented by an e-circle contained in $\overline{U}$, which is tangent to ∂U, say at A. The half-plane can be seen as the conformal image of U which transforms A into P^∞ which plays the rôle of a privileged point. Then the set of *horocycles* which touch ∂U at A is transformed into the horizontal lines of P. The orthogonal curves to these horocycles are true hyperbolic lines which all pass through A. These lines are transformed into the vertical lines of P. On the other hand, the disk model gives a more uniform image of the hyperbolic plane. It may be more useful for some purposes. As an example, in order to devise cellular automata in which the definition requires homogeneity in the spatial distribution of the cells, the disk model seems to be somewhat more appropriate. Below, in Figure 1 and in Figure 3, the same object is represented in both contexts. This is also the case for Figures 5 and 6. Thus, the reader may consider the advantages of each representation.

Consider a polygon S. The set of polygons generated by S consists of S, all the reflections of S in its sides and, recursively, in the sides of its images. In the theory of tilings, the set of polygons generated by S is a *tiling* if and only if the interiors of two distinct polygons either coincide or do not intersect, and if any point of the plane is in the closure of at least one polygon of the set. In this case, we say that the tiling is generated by S. We shall also say that the tiling is the *grid* defined by S.

A well known theorem proved by Henri Poincaré in the eighteen-eighties states that there are infinitely many tilings of the hyperbolic plane that are generated by a triangle, see for instance Caratheodory [2] for an elementary proof. It is not true that any triangle generates a tiling of the hyperbolic plane, and Poincaré's theorem includes a sufficient condition on the angles of the triangle to ensure the tiling property: the angles are of the form $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$ and the inequality $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ is satisfied. This latter inequality says precisely that the sum of the interior angles of the triangle is less than π, which is true for any triangle in $I\!H^2$.

As a consequence, in the hyperbolic plane, there are *always* polygons with regular sides for which the vertex angle is a right angle at every vertex, for any number of sides $s \geq 5$: take the triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{s}$; setting $2.s$ copies of these triangles around a common vertex by their $\frac{\pi}{s}$ angle constitutes a rectangular regular s-gon. Another consequence of the theorem which is also a general property of the hyperbolic geometry, is that all such polygons with the same number of sides are isometric. Now, assume that we display these polygons for all the values of $s \geq 5$, the number of sides, with a vertex in a fixed point of the hyperbolic plane, for instance in O, the centre of ∂U, and with the two edges at that point on the same orthogonal lines. As suggested in Figure 1, below, we see that these polygons tend to a *limit* which has an infinite number of sides. It looks like a polygon, but it is not a finite figure: it has a point at infinity. In Figure 3, we give the representation of that figure in the half-plane model.

In the report, we call such an object an *infinigon*. As a consequence of the definition that we shall give later, there are *basic* infinigons and there are *open* ones. Basic infinigons have a single point at infinity, while open ones have a whole continuum of points at infinity: an interval of the points at infinity of the hyperbolic plane. An indirect reference to *basic infinigons* can be found in [3,4]. In these papers, the author gives no name to these objects that are implicitly defined by the tilings which they generate and which are characterised as a *limit* case. This gives the basic infinigons for which the interior angle is compatible with a tiling of $I\!H^2$, assuming that this compatibility is enough to actually tile $I\!H^2$. Also in [3,4], the author only indicates the connection of the basic infinigons with horocycles.

We shall start with a simple construction that will allow us to introduce infinigons in a simple and elementary way, with full generality, and that will directly lead to the separation between *basic* and *open* infinigons. Later, after a characterization of the finite polygons, we define the infinigons. In the next section, we give another purely geometric way to define infinigons: via linear pencils of lines. In the last section, we characterise the infinigons from which it is possible to define a grid in the hyperbolic plane. We give an algorithm to construct such tilings.

Γ

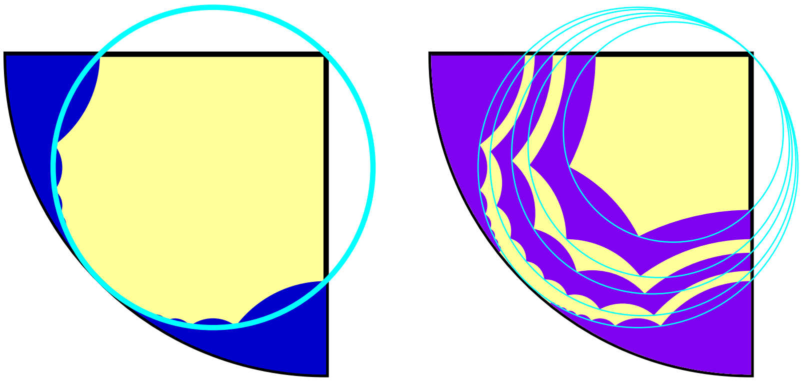

Figure 1 Left: the basic infinigon with angle $\frac{\pi}{2}$;
Right: some of its finite sided approximations.

1. Basic features.

1.1 The basic construction

In the case of the disk model, fix two orthogonal diameters of U. One is called horizontal and the other one vertical. Following [6] and [7], the cone of positive coordinates will be represented in the figures by the south-western quarter which is associated to these diameters.

In order to define the infinigons, we consider the following process in $I\!H^2$: given two points x_{n-1} and x_n and the angle α, we first construct the h-line β_n which passes through x_n and which makes an angle of $\alpha/2$ with the h-line $x_{n-1}x_n$, clockwise. Define x_{n+1} to be the image of x_{n-1} by the reflection in β_n. We repeat the process indefinitely, starting from x_0 and from x_1. In the case of the disk model, we set $x_0 = 0$ and we fix x_1 on the horizontal axis, towards the positive abscissas, at the euclidean distance x from x_0, with $x \in [0, 1[$. In the case of the half-plane model, x_0 is any point at the euclidean distance 1 from ∂P, but x_0x_1 is chosen in such a way that it makes an angle of $\alpha/2$ with the vertical line which passes through x_0. We denote by ℓ the euclidean distance between x_0 and x_1.

Notice the following property of the half-plane model: if we fix x_0 at an arbitrary euclidean distance x from ∂P, we obtain a new figure that is similar to the one that is obtained with x_0 at distance 1, provided that we take $x\ell$ instead of ℓ. In the hyperbolic plane, the new figure is the isometric image of the previous one under a shift.

Theorem 1. – *For all α and x, points x_n which are obtained by the previous construction belong to a euclidean circle or a euclidean line, which we shall refer to as Γ.*

In the disk model, Γ is always a circle with $\dfrac{x}{\cos(\frac{\alpha}{2})}$ as a diameter.

In the half-plane model, Γ is an e-line if and only if the equality $\ell = 2x\mathrm{cotan}(\dfrac{\alpha}{2})$ holds, and when this is not the case, Γ is an e-circle with its centre on the vertical line which passes through x_0 with $\dfrac{\ell}{\cos(\frac{\alpha}{2} + \arcsin(\frac{\ell}{2x}\sin(\frac{\alpha}{2})))}$ as diameter.

Moreover, in both cases, curvilinear abscissae of the x_n's on Γ starting from x_0 toward x_1 are increasing.

The proof is illustrated by Figure 2 and Figure 3, below.

First assume that we are in the disk model U.

Let δ_1 be the e-line which goes through x_1 and that makes an angle of $\dfrac{\alpha}{2}$ with the e-line which supports x_0x_1. Consider the e-circle Γ which passes through x_0, x_1 with its e-centre on the e-line δ_1. By construction, Γ is orthogonal to β_1, the h-line which is tangent to δ_1 at x_1. Consequently, Γ is invariant under the inversion with respect to β_1 or, which is the same, in the reflection in β_1. Accordingly, if β_0 is the h-line which passes through x_0 and which makes an angle of $\dfrac{\alpha}{2}$ with the h-line x_1x_0, Γ is also orthogonal to β_0: if δ_0 is the symmetric image of δ_1 by the e-bisector of the e-segment x_1x_0, δ_0 is tangent to β_0. Accordingly, the centre of Γ is on the e-bisector of the e-segment x_0x_1. We easily deduce the computation of the diameter of Γ from this information.

Notice that the point x_2 is the image of x_0 under the reflection in β_1. The h-line x_1x_2 is easy to construct. By definition, the radical axis of a point y inside U with respect ot ∂U is an e-line which is the set of the centres of the circles which go through y and which are orthogonal to ∂U: hence, it is the set of the centres of the e-circles of the h-lines which go through y. As a result of elementary geometry, this set is an e-line. And so, the e-centre of the e-circle which supports the h-line x_1x_2 is at the intersection of the radical axis D_1 of x_1 with respect to ∂U and of the radical axis D_2 of x_2 with respect to ∂U. This explains the construction that is performed in Figure 2.

Now, this argument can be repeated word by word with x_n, x_{n+1} and x_{n+2}. It is enough to assume that Γ is orthogonal to β_n and to β_{n+1} and that x_{n+2} and β_{n+2} are obtained respectively from x_n and β_n by the reflection in β_{n+1} as an induction hypothesis. The proof of the induction step is contained in the proof of the basis case that we have given above.

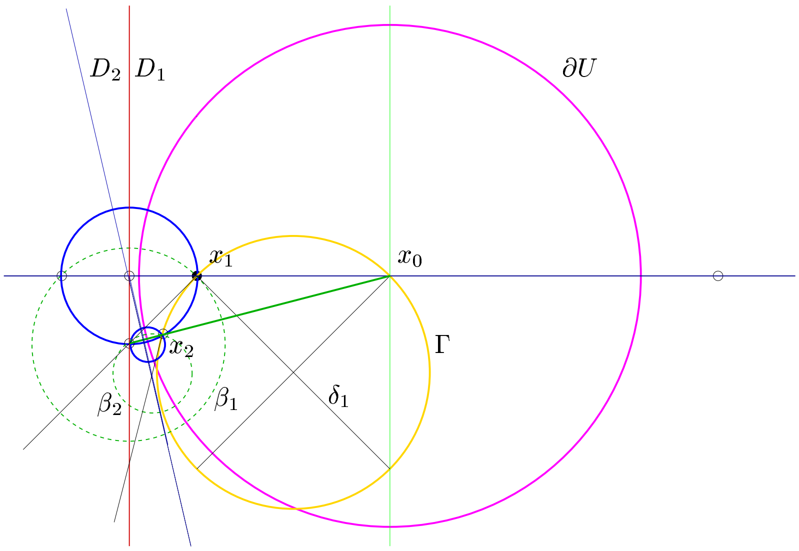

Figure 2 The basic construction in the disk model

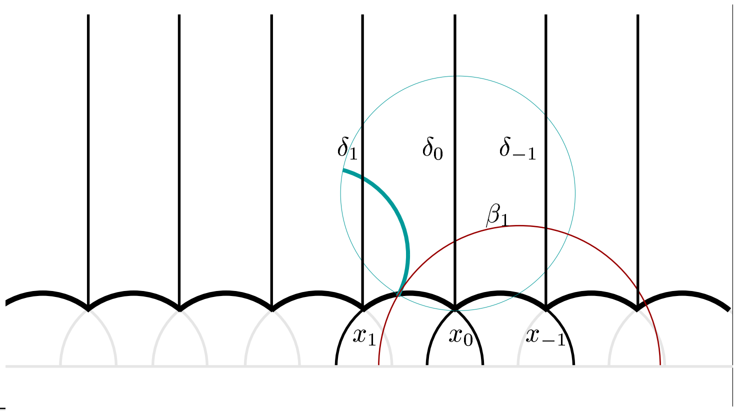

Figure 3 The representation in Π of a basic infinigon. Notice the lines δ_i which are bisectors of the interior angles at the vertices of the infinigon. The first step of a construction for a line which does not give rise to an infinigon is also represented.

Now, assume that we are in the half-plane model.

The above construction can be repeated word by word except in one case, when x_1 and x_0 are on the same e-horizontal line and when δ_1 is a vertical line.

In this case, which is illustrated in Figure 3, the hyperbolic reflections are identical with the euclidean reflection in the same lines, and it is easy to see that all x_n's remain on the same e-line Γ which is also orthogonal to all β_n's: in this case, β_n is the vertical line which passes through x_n.

In this case, if β is the angle of the h-line x_0x_1 with the e-line x_0x_1, we have that $\ell = 2r.\sin\beta$, where r is the radius of the e-circle that supports the h-line x_0x_1. Now, $x = r.\sin\dfrac{\alpha}{2}$. Also, $\beta + \dfrac{\alpha}{2} = \dfrac{\pi}{2}$ and so, $\ell = ||x_0x1|| = 2x\mathrm{cotan}(\dfrac{\alpha}{2})$.

When $\ell \neq 2x\mathrm{cotan}(\dfrac{\alpha}{2})$, then Γ is an e-circle. This time, if R is the radius of Γ, we get that $\dfrac{\ell}{2} = R.\cos(\dfrac{\alpha}{2}+\beta)$, which gives $R = \dfrac{\ell}{2\cos(\frac{\alpha}{2}+\beta)}$. Replacing β by the relations that we get between ℓ, r and β, we obtain the expression which is formulated in the statement of the theorem for the diameter of Γ.

By construction, the length of the arc $x_{n+1}x_{n+2}$ on Γ is greater than the length of the arc x_nx_{n+1}, either the length is taken in the euclidean sense or in the hyperbolic one. Also, the line β_n is the inner bisector of the angle $x_{n-1}x_n, x_nx_{n+1}$. ■

1.2. Polygons.

Assuming that we are considering the disk model, Theorem 1 entails that if we have $\dfrac{x}{\cos(\frac{\alpha}{2})} < 1$, all x_n's remain in $I\!H^2$, and so, circle Γ is a true h-circle. We have the following property:

THEOREM 2.*a*. – *For $x < \cos\left(\dfrac{\alpha}{2}\right)$ the sequence $\{x_n\}$ defines a regular polygon if and only if $x = \sqrt{\dfrac{\cos^2\frac{\pi}{k} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\pi}{k}}}$ for some k with $\dfrac{\alpha}{2} + \dfrac{\pi}{k} < \dfrac{\pi}{2}$. When x is not equal to $x = \sqrt{\dfrac{\cos^2\frac{h\pi}{k} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{h\pi}{k}}}$ for some h and k with $\dfrac{\alpha}{2} + \dfrac{h\pi}{2k} < \dfrac{\pi}{2}$, the sequence is an infinite polygonal line which is dense in a hyperbolic annulus,*

where a hyperbolic annulus is the complement in a closed hyperbolic disk of a closed disk with the same center but a smaller radius.

Proof. Consider the h-bisector of the line x_1x_0, which cuts that line in point I with abscissa v. The bisector is supported by an e-circle (Σ) with equation $(X-z)^2+Y^2 = z^2-1$, where $(z,0)$ is the centre of the circle. The value of the radius comes from the orthogonality of (Σ) with $\partial U : z(z-x) = (z-v)^2$ from which we get that $zx = 1$. The equation of (Σ) reduces to the following form: $X^2+Y^2-2zX+1 = 0$. Let $Y = aX$, with $a = \tan\dfrac{\alpha}{2}$, the line through O with angle $\dfrac{\alpha}{2}$ with x_1x_0. The intersection in U with (Σ) is given by $X = \dfrac{z-\sqrt{z^2-(1+a^2)}}{1+a^2}$, with the condition $z > \sqrt{1+a^2}$ for an intersection in U, which gives $x < \cos\dfrac{\alpha}{2}$.

Under that condition, let H be the point of intersection in U of the line $Y = aX$ with (Σ). Let θ be the angle of $Y = aX$ with (Σ) at H. This is given by the euclidean angle of $Y = aX$ with the tangent to (Σ) at H. The standard computation with the scalar product of two vectors divided by the product of their norms provides us with $\cos\theta = \dfrac{az}{\sqrt{1+a^2}\sqrt{z^2-1}}$. Coming back to α and x we get $\cos\theta = \dfrac{\sin\frac{\alpha}{2}}{\sqrt{1-x^2}}$.

The formula of the theorem easily ensues from the above computation: the line is a regular (convex) polygon if and only if $2\theta = \dfrac{2\pi}{k}$ for some $k \geq 3$. Also, the polygonal line is dense in the annulus if and only if θ is irrational in π. Notice that the positivity of the argument of the square root entails that $\sin\frac{\alpha}{2} < \cos\theta$, which means that $\frac{\alpha}{2} + \theta < \frac{\pi}{2}$, which means that the triangle IHO is a true triangle of the hyperbolic plane. ■

Assume that we are now in the half-plane. It is easy to see that Γ is an e-circle that is contained in P if and only if $\ell < 2\mathrm{cotan}(\frac{\alpha}{2})$. In that case,

THEOREM 2.*b*. – *For* $\ell < 2\,\mathrm{cotan}\,\frac{\alpha}{2}$, *the sequence* $\{x_n\}$ *defines a regular polygon if and only if* $\cos\frac{\pi}{k} = \sin(\frac{\alpha}{2}+\gamma)\sqrt{\frac{\sin\frac{\alpha}{2}}{\sin(\frac{\alpha}{2}+2\gamma)}}$, *with* $k \geq 3$ *and where* γ *is the angle of the tangent of the h-line* x_1x_0 *with the e-line* x_1x_0, *which entails that* $\ell = \frac{2x}{\sin\frac{\alpha}{2}}\sin\gamma$. *For each value of* $k \geq 3$, *there is precisely one value of* γ *for which the above equality holds. When this is not the case, the sequence is an infinite polygonal line that is dense in a hyperbolic annulus,*

where a hyperbolic annulus in the half-plane is defined in the same way as it is in the disk model.

Proof. Consider the h-bisector of the line x_1x_0. It is supported by an e-circle (Σ) with equation $(X-a)^2+Y^2 = R^2$, where $(a,0)$ are the e-coordinates of the centre of (Σ) and R is its radius. The circle (Σ) cuts the vertical line δ_0 which passes through x_0 in a point J with coordinates $(0,\sqrt{R^2-a^2})$, see Figure 4. The tangent to (Σ) at J has $(-\sqrt{R^2-a^2}, -a)$ as coefficients and so, if θ is the angle of that tangent with δ_0, we have that $\mathrm{cotan}\,\theta = \frac{a}{\sqrt{R^2-a^2}}$ which is equivalent to $\cos\theta = \frac{a}{R}$. Let $-\xi_1$ and η_1 the coordinates of x_1. As x_1, x_0 and a are on the same e-line, and as we may assume that $x_0 = (0,1)$, we get that $\eta_1 = \lambda$ and $a+\xi_1 = \lambda a$ and so, $a = \frac{\xi_1}{\eta_1 - 1}$. From the fact that x_1 is the image of x_0 in the reflection in (Σ), we have $(a^2+1)((a+\xi)^2+\eta^2) = R^4$ and so, $R = \sqrt{\lambda}\sqrt{a^2+1}$. An easy computation leads to $\cos\theta = \frac{\xi_1}{\ell\sqrt{\lambda}}$, because $\ell = \sqrt{\xi_1^2+(\eta_1-1)^2}$.

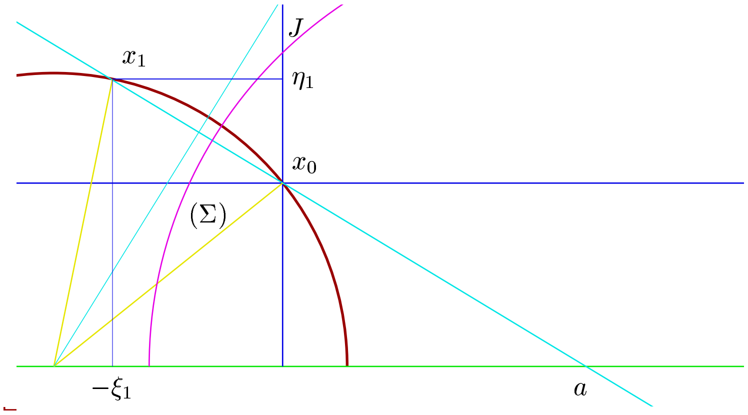

Figure 4 The bisector of the h-segment x_0x_1 and its neighbouring.

Now, let r be the radius of the e-circle which supports the h-line x_1x_0. It is not difficult to obtain that $\eta_1 = r\sin(\frac{\alpha}{2}+2\gamma)$, that $r = \dfrac{1}{\sin(\frac{\alpha}{2})}$ and that:

$$\xi_1 = r\cos(\frac{\alpha}{2}) - r\cos(\frac{\alpha}{2}+2\gamma) = 2r\sin\gamma\sin(\frac{\alpha}{2}+2\gamma).$$

Now, as $\ell = 2r\sin\gamma$, we obtain that:

$$\cos\theta = \frac{\sin(\frac{\alpha}{2}+\gamma)}{\sqrt{\eta_1}} = \sin(\frac{\alpha}{2}+\gamma)\sqrt{\frac{\sin\frac{\alpha}{2}}{\sin(\frac{\alpha}{2}+2\gamma)}}.$$

A calculus of the derivative of the expression of $\cos\theta$ with respect to γ shows that this derivative is

$$(\cos\theta)\Big(\mathrm{cotan}(\frac{\alpha}{2}+\gamma) - \mathrm{cotan}(\frac{\alpha}{2}+2\gamma)\Big).$$

This difference of cotangents is positive as far as γ ranges in the interval $]0, \frac{\pi}{2}-\frac{\alpha}{2}]$. And so, $\cos\theta$ is an increasing function of γ and θ is a decreasing function of γ.

Later on, the proof goes on as for the disk model. ■

Under the assumptions of theorem 1, define by α_n the angle between x_nx_{n+1} and $x_{n+1}x_{n+2}$. Then the inner h-bisectors of α_n's meet in the hyperbolic center of Γ.

Indeed, this is a consequence of the above construction: J is the hyperbolic centre of Γ. As (Σ) and β_1 pass through J by construction, by reflection in β_1, β_2 also passes through J and this is enough to be kept during the induction.

3. Infinigons.

3.1. Construction of the infinigons.

Consider the process that is described by Theorem 1 and assume that we are in the unit disk U.

When $x \geq \cos\big(\dfrac{\alpha}{2}\big)$, if we consider the unit disk U, we obtain an infinite polygonal line which does not remain within a circle.We define a symmetric part of the process. In the disk, let L_α be the e-line with equation $Y = X\tan(\alpha)$. Its trace in U is a h-line which passes through O and which makes an angle of α with x_0x_1. Let Δ_α be the h-bisector of that angle. The reflection in Δ_α transforms the x_i's into x'_i's with $x'_0 = x_0$. In the half-plane, we consider the reflection in δ_0, see Figure 3. We call the infinite polygonal ligne defined by the x_i's and the x'_i's an *infinigon*.

Consider again the unit disk U. The sequence of x_n's converges to a point at infinity Ω_ℓ. The sequence of x'_n converges to a point at infinity Ω_r, where Ω_r and Ω_ℓ are the points of intersection of Γ, the circle of Theorem 1, with ∂U. Indeed, the sequence of x_n's remains in $\overline{U}$. As the curvilinear abscissa of the x_n's on Γ increases, the x_n's converge to a limit in $\overline{U}$. The limit cannot be a point of $I\!H^2$, because $||x_nx_{n+1}||_h = ||x_0x_1||_h$, where $||\,||_h$ is the hyperbolic distance, and so the limit is on ∂U. As Γ is closed it is also on Γ.

When $x = \cos\big(\dfrac{\alpha}{2}\big)$, it is easy to see that $\Omega_\ell = \Omega_r = \Omega$. In that case, the inner h-bisectors of angles α_n meet in Ω which is the point of tangence of Γ with ∂U, and Γ is a horocycle. Again, in the half-plane, we find that the horocycle is the e-line defined by the x_i's and x'_i's.

On the other hand, by construction of the sequence of x_n's and x'_n's, the inner bisectors of angles α_n are axes of reflection for the infinigon. If the h-bisector were again to cut the polygonal line, one of the two delimited parts would entirely fall into $I\!H^2$, within a large enough closed disk, which can be made invariant by reflection in the bisector. By the same reflection, the whole line would fall within that disk, which is contrary to our hypothesis. As Ω is the single point at infinity through which the inner h-bisectors may pass without again meeting the sides of the infinigon, these h-bisectors pass through Ω.

In this particular case, we say that the infinigon is a *basic* infinigon.

Consider the case that $\ell = 2\text{cotan}\dfrac{\alpha}{2}$, assuming that we are now in the half-plane P.

Figure 3 above displays a representation of the infinigon in this setting. The construction is more simple in the case of the basic infinigon. Indeed, we know that Γ is tangent to the unit circle on a point P at infinity: it is a *horocycle*. If we take a conformal mapping of the unit disk that transforms it into the half-plane by sending P to infinity, the image of Γ is a horizontal line of the half-plane as we already noticed, say η. Now, consider a family of vertical lines that are equidistant in the euclidean sense. Let x_n, $n \in \mathbb{Z}$, be the intersections of these lines with η. It is known that $x_n x_{n+1}$ are equidistant in the hyperbolic sense. The important thing to keep in mind is that a horizontal euclidean line *is not* a hyperbolic line and that if η' is another horizontal e-line that cuts the considered verticals in $x'_n x'_{n+1}$, the quadrangle $x_n x_{n+1} x'_{n+1} x'_n$ *is not* a rectangle. In particular, $||x_n x_{n+1}||_h < ||x'_n x'_{n+1}||_h$ if η' is higher than η in the euclidean meaning.

Let us go back to the disk model where the following property is seen more easily.

The basic infinigon of angle α can also be seen as follows. Let $\Pi_{\alpha,k}$ be the sequence of regular polygons with inner angle α and with k sides, k such that $\dfrac{1}{k} + \dfrac{\alpha}{2} < \dfrac{1}{2}$. We may assume that each $\Pi_{\alpha,k}$ has O as a vertex and one of its sides on the positive horizontal radius. Fix $N > 0$, and let $S_{N,k}$ be the N^{th} vertex of $\Pi_{\alpha,k}$, counting anti-clockwise from O. For $N \leq 0$, say that $S_{N,k}$ is the $|N|^{\text{th}}$ vertex of $\Pi_{\alpha,k}$, counting clockwise from 0. Then the sequence of $S_{N,k}$'s for $k \geq |N|+1$ converges to x_N. We say that the $\Pi_{\alpha,k}$'s *visually* converge to the basic infinigon.

When $x > \cos\left(\dfrac{\alpha}{2}\right)$, we say that the infinigon is *open*.

THEOREM 3. – *For $x > \cos\left(\dfrac{\alpha}{2}\right)$, or for $\ell > 2\,\mathrm{cotan}\,\dfrac{\alpha}{2}$, the basic sequence and its reflection in L_α or δ_0 defines an open infinigon. Thus, there are continuously many open infinigons.*

In the case of an open infinigon, the x_n's also converge to a point at infinity which is one of the two points of intersection of Γ with ∂U : the point which is in the same e-half-plane as x_n's with respect to L_α. The inner bisectors of the angles α_n do not meet the limit of the sequence of x_n's. If that were the case, considering a reflection in an inner bisector, by a similar argument to that of the proof of theorem 2, we would obtain that the infinigon would be a basic infinigon. Accordingly, two inner bisector do not meet in $\overline{U}$ and so, as their euclidean supports are orthogonal to ∂U, as e-circles they do not meet at all. Consequently one contains the other one. But now, it is easy to see that the corresponding e-circles converge to one of the point of intersection of Γ with ∂U.

THEOREM 4. – *When $x = \cos\left(\dfrac{\alpha}{2}\right)$, or when $\ell = 2\,\mathrm{cotan}\,\dfrac{\alpha}{2}$, the basic* *sequence together with its reflection in L_α or in δ_0 defines a basic infinigon. Accordingly, there are continuously many basic infinigons and for each α, the basic infinigon is the visual limit of the regular polygons with angle α.*

We conclude this section with a remark of group theory: consider the bisectors of the angles at a vertex, say Δ_i for $i \in \mathbb{Z}$. Consider also the bisectors of the sides, say Π_i for $i \in \mathbb{Z}$. We can assume that the indices are chosen in such a way that the same side is concerned by both Δ_i and Π_i. Then, the composition of the reflection in Δ_i with the reflection in Π_i is an *ideal rotation* because the two lines are parallel, which we can restate as: they meet at a point at infinity. Denote this ideal rotation by ρ_i. Then ρ_i transforms any side s_j of the infinigon into the side s_{j+1}. And so, the infinigon appears to be the orbit of any of its vertices with respect to that ideal rotation. This is particularly clear on Figure 3 which refers to the half-plane model. We also obtain that all ρ_i's appear to be the power of a fixed ρ_0 and so, these rotations constitute a cyclic group isomorphic to $\mathbb{Z}$. In the case of a basic infinigon, the orbit has a limit point which is the point at infinity of the infinigon. In a case of an open infinigon, there are two limit points which are also points at infinity of the infinigon and the orbits are no more supported by horocycles but by arcs of e-circles.

3.2. Infinigons and pencils of lines.

In this sub-section, we consider infinigons from the point of view of *pencils of lines*, which also plays an important rôle in hyperbolic geometry. Remind that, a *pencil of lines* is a family of lines which is indexed by the real numbers in a *linear* way. There are three kinds of pencils in the hyperbolic plane, depending on the mutual intersection between two lines of the family.

Either all lines of the pencil meet in the same point A of the hyperbolic plane, or all the lines of the pencil meet in the same point A at infinity, or all lines of the pencil are pairwise non-secant and there is another line p which is the common perpendicular of any two lines of the pencil.

Consider the case when all the lines of the pencil share the same point A of the hyperbolic plane. In that case, the orthogonal curves to the lines of the pencil are the circles included in U whose h-centre is A.

In the case when all the lines of the pencil share the same point A at infinity, the orthogonal curves to the lines of the pencil are the horocycles that are tangent at A with ∂U. The representation is particularly simple in the half-plane model: we take A as P^∞, the point at infinity of P that does not belong to ∂P. Then, the horocycles in A are the horizontal e-lines. Notice that the vertical lines that are their orthogonal curves represent the axes of the ideal rotations which we mentioned in the previous section.

Lastly, in the case when all the lines of the pencil are perpendicular to the same line, say p, the orthogonal curves are the intersection with U of e-circles which all pass through A and B, the points at infinity of p. This is also the case if we consider the same case in P.

In all cases, the orthogonal curves are arcs of e-circles, possibly complete arcs.

Let us fix two lines ℓ_0 and ℓ_1 of the pencil. Now, define σ_n to be the reflection in ℓ_n, where ℓ_n is inductively defined by $\ell_{n+2} = \sigma_{n+1}(\ell_n)$, for all $n \in \mathbb{Z}$. This defines a subset of the pencil that we call an *infinigonal pencil.*

We have the following theorem:

THEOREM 5. – *If ℓ_n is an infinigonal pencil, the trace of the pencil on a curve that is orthogonal to the pencil, say Γ, is one of the following sets:*

- *(i), a dense susbset of points of Γ, or the vertices of a regular polygon which is inscribed in Γ when the lines of the pencil meet in a point of the plane;*
- *(ii), the vertices of a basic infinigon I, and Γ is a horocycle with the limit point of I as a point at infinity for Γ;*
- *(iii), the vertices of an open infinigon I, and Γ is an e-circle that passes through the limit points of I.*

Proof. We consider at the same time, the three cases (i), (ii) and (iii) when, respectively, the lines of the pencil either meet in a point A in the hyperbolic plane or at infinity, or they are orthogonal to a common perpendicular p.

Consider Γ a curve that is orthogonal to the pencil. Γ is either a circle of the hyperbolic plane, centered in A in case (i), or a horocycle in A in case (ii) or an e-circle which passes through two distinct points at infinity, A and B in case (iii). Let $C_n = \Gamma \cap \ell_n$. Then, by the symmetries used in the construction, ℓ_{n+1} is the bisector of the angle $C_{n+2}C_{n+1}, C_{n+1}C_n$. Accordingly, the sequence of points C_n is constructed according to the basic process that is defined in the first section of the paper. And so, the theorem is a direct consequence of Theorem 1. ∎

Notice that the theorem provides us with a new way fo constructing infinigons.

An alternative way consists in noticing that the *ends* of the lines ℓ_n's, *i.e.* the points at infinity of the lines, say I_n, satisfy the same symmetry relation: $I_{n+2} = \sigma_{n+1}(I_n)$. The points I_n satisfy the relation $I_n = \partial U \cap \ell_n$ or $I_n = \partial P \cap \ell_n$. Indeed, they can be generated by fixing two points, I_1 and I_0 at infinity, and then, the lines with ends A and I_n are easily generated.

4. The grids.

In this section, we focus our attention on the grids that can be defined by the infinigons. Indeed, in that case, the standard definition can be applied word by word: starting from a given infinigon, we duplicate it by the reflections in its sides, and then we repeat the process recursively to the newly obtained infinigons. It is clear that this process does not give rise to a grid for *any* infinigon. In particular, it is plain that the angle at a vertex must be a of the form $\dfrac{2\pi}{k}$, with $k \geq 3$.

The converse is true:

THEOREM 6. – *Let x be a real number in $]0,1[$ with $x \geq \cos\left(\dfrac{\alpha}{2}\right)$ or fix a real number ℓ such that $\ell \geq 2\,\mathrm{cotan}\,\dfrac{\alpha}{2}$. The infinigons which are associated to α and x, or to α, x and ℓ define a grid if and only if $\alpha = \dfrac{2\pi}{k}$, with $k \in I\!N$, $k \geq 3$. When the grid exists, it is unique.*

The technique that we give in [6] and [7] can be re-used here to obtain an elementary proof of the theorem. In Figure 5, we give an illustration representing the infinigon with right angles in the half-plane model. In Figure 6, we give the representation of the same object in the disk model.

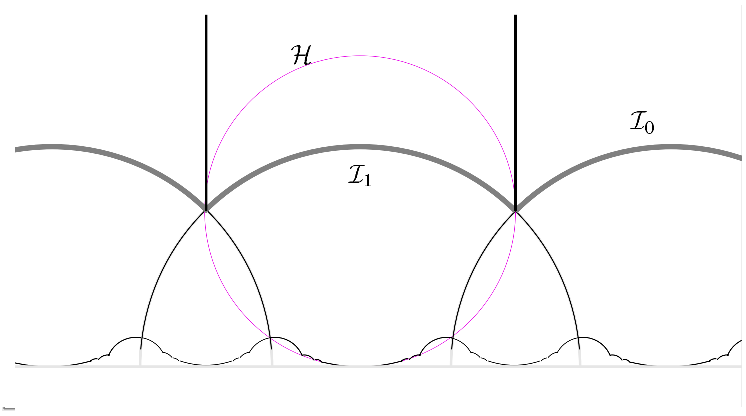

Figure 5 A view on the infinigrid with right angles in the half-plane model.
Notice that the infinigons which are distinct from $\mathcal{I}_0$ are circumscribed by horocycles that exactly look like horocycles in the disk model. See the example of $\mathcal{I}_1$ which is circumscribed by $\mathcal{H}$.

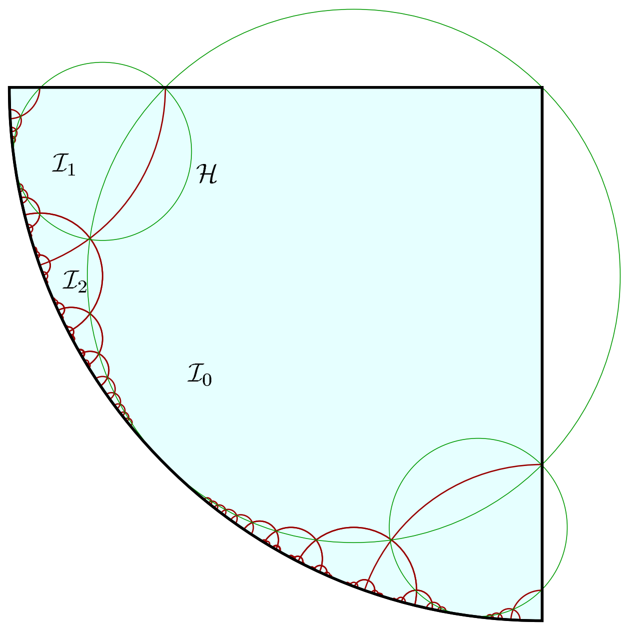

Figure 6 A view on the infinigrid with right angles in the disk model.
Now all infinigons are circumscribed by e-circles which are *horocycles*.

We first define a recursive splitting of the hyperbolic plane from which the existence of the grid can be inferred. Next, it is possible to build a tree which can easily be put into a one to one correspondence with the infinigons of the grid. The tree is a sub-graph of the dual graph of the grid. It has the same vertices as the graph but a certain number of edges are missing: we represent them by dotted lines in the figures below.

Indeed, let $\mu\nu$ be an edge between nodes μ and ν. This edge is interpreted as the reflection in the side that is common to the infinigons which are attached to μ and ν respectively. Besides, if one of the nodes is defined to belong to the $n^{\rm th}$ generation, the other belongs, by definition, to the $n{+}1^{\rm st}$ generation. This means that the new graph is directed. In order to obtain a tree, we delete edges which would replicate an infinigon of the $n^{\rm th}$ generation onto another infinigon of the next generation. There are continuously many ways to remove the edges: this generalises what has been done in [7] for the rectangular pentagonal grid.

4.1. Building the splitting of $I\!H^2$.

Our arguments will be illustrated in the unit disk in a more convincing way.

First, we split $I\!H^2$ into k angular sectors with O as a common vertex. Let $\mathcal{S}$ be such a sector, see Figure 7, where we sketchily represented the case when k is even on the right hand side of the figure and the case when k is odd on its left hand side. Figures 5 and 6 should convince the reader why we adopt a stylistic representation of the grid for Figure 7 and for those which follow.

Now, notice that $\mathcal{S}$ contains an infinigon, say $\mathcal{I}$, in which the angle of the sector is an interior angle of the infinigon. We say that $\mathcal{I}$ is the *leading infinigon* of the sector. The closure of $\mathcal{S}\backslash\mathcal{I}$ in $I\!H^2$ can be characterised as follows. Let A_0 be the vertex of the leading infinigon which lies on the left hand border of $\mathcal{S}$ at the appropriate distance from O. The side A_0A_1 is continued by a h-half line which delimits, with the left hand border of $\mathcal{S}$, an angular sector $\mathcal{P}_0$ with angle $\pi-\dfrac{2\pi}{k}$. Let $p=\lfloor\dfrac{k}{2}\rfloor$.

Assume that $\mathcal{P}_0$ is already split. Displacement along the h-line A_0A_1 with amplitude A_0A_1 followed by an appropriate rotation transforms $\mathcal{P}_0$ into an isometric sector, say $\mathcal{P}_1$, which can also be split as $\mathcal{P}_0$ is split. We proceed in the same way with all the sides of the infinigon which are on the same side of the bisector of angle O as A_0. By reflection in that bisector, another analogous splitting is obtained for the other sides of the infinigon.

When k is even, $\pi-\dfrac{2\pi}{k}=(p{-}1)\dfrac{\pi}{p}=(p{-}1)\dfrac{2\pi}{k}$ and so, $\mathcal{P}_0$ is the union of $p{-}1$ angular sectors which are isometric to $\mathcal{S}$.

When k is odd, $\pi-\dfrac{2\pi}{k}=(2p{+}1)\dfrac{\pi}{k}-\dfrac{2\pi}{k}=(p{-}1)\dfrac{2\pi}{k}+\dfrac{\pi}{k}$. And so, as we see on the right hand side of Figure 7, we first obtain in A_0 an angular sector of angle $\dfrac{\pi}{k}$ that is isometric to one half of $\mathcal{S}$. Then, there is room in $\mathcal{P}_0$ exactly for the union of $p-1$ sectors which are also isometric to $\mathcal{S}$. Let $\mathcal{S}_\ell$ and $\mathcal{S}_r$ be, respectively, the left hand and the right hand halves of $\mathcal{S}$. Accordingly, the half-sector which we distinguished in $\mathcal{P}_0$ is isometric to $\mathcal{S}_r$. In $\mathcal{S}_r$, the right hand half of $\mathcal{S}$, let $\mathcal{P}_{-1}$ be the symmetric image of $\mathcal{P}_0$ with respect to the bisector of the angle in O. We split $\mathcal{P}_{-1}$ symmetrically: the half sector which is raised by the splitting is the isometric image of $\mathcal{S}_\ell$.

Splitting provides us with one infinigon per sector, the leading one. Later, we proceed recursively in each one of the obtained sectors. Each iteration of the process increments the order of generation. Notice that in a sector, we have both kinds of half sectors and that this is also the case in a half sector because it contains sectors which are all isometric to the initial one. It is enough to define the tree in a sector and then in a half sector for the odd case in order to define the whole tree, by induction. We can notice that each continuation of the process corresponds to isometric images of the initial sector or of one of its halves.

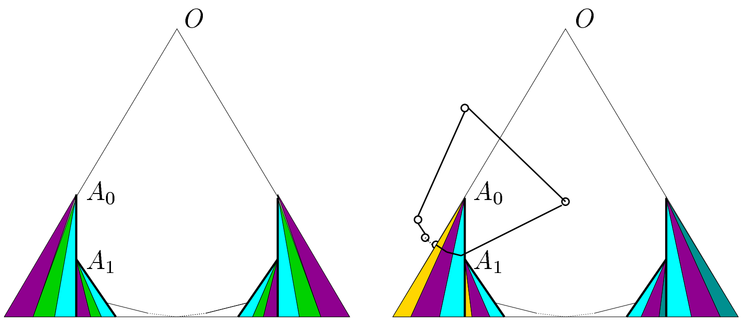

Figure 7 Splitting a sector of angle $\dfrac{2\pi}{k}$

Splitting defines the grid uniquely. As in [6], it is easy to prove that the euclidean diameter of an angular sector tends to 0. The diameter of an angular sector from O is $\max(1, 2\sin\dfrac{\alpha}{2})$. Let r be the distance from A_0 to ∂U. Then the diameter of an angular sector from that A_0, with one border on the line OA_0, is $r.\max(1, 2\sin\dfrac{\alpha}{2})$. But that sector contains the h-isometric shift of $\mathcal{S}$ along OA_0 with amplitude OA_0. We can then reproduce the computation which is given in the lemma 3 of [6]: the computation provides us with the abscissas α_n of points A_n on the same h-line from O, with $A_{-1} = O$ and such that A_{n+1} is the translation of A_{n-1} along the line of amplitude $A_{n-2}An$. It is shown that $\alpha_n \to 1$ as n tends to infinity. When we repeat the translation that is considered in splitting, we have an easy upper estimate of r by $1-\alpha_n$. And so, as announced, the e-diameters of the h-isometric images of $\mathcal{S}$ tend to 0 as the order of generation in the splitting tends to infinity.

Now we obtain the tiling property: any point of $I\!H^2$ falls in at least one region defined by splitting. If it is an infinigon, we are done. If no infinigon contains the point, say A, then at each generation, the point is in a sector which is part of the complement of the leading infinigon in the region where it belongs at the previous generation. And so, this defines a sequence of sectors $\mathcal{S}_n$ (possibly half sectors among them) which are pairwise embedded and whose e-diameters tend to 0. And so, this sequence defines a point in $\overline{U}$, namely A. But this point cannot be in $I\!H^2$. Indeed, let B_n be the vertex of $\mathcal{S}_n$ that corresponds to O in Figure 7. It is plain that $||B_nA||_e$ tends to 0 as n tends to infinity. By construction, $||B_nB_{n+1}||_h \geq ||OA_1||_h$, so long as $\mathcal{S}_n$ is isometric to $\mathcal{S}$ or to one of its halves. And so, the sequence of B_n's cannot converge to any point of $I\!H^2$. Accordingly, the limit is on ∂U, a contradiction with the assumption that $A \in I\!H^2$.

Notice that this proof of the tiling property is not constructive. Indeed, it cannot be so, due to the fact that we conclude that A belongs to the *closure* of some infinigon. What we can prove is that for each point A of $I\!H^2$, there are two neighbouring infinigons $\mathcal{I}$ and $\mathcal{J}$ such that $P \in \mathcal{I} \cup \mathcal{J}$ and we cannot say, in general, to which member of this union the point belongs precisely.

4.2. Building the tree.

Our goal is to devise finitely many *algorithmic*-like *rules* that allow us to build the tree by their *automatic* application. Each node of the tree can be reached within a finite number of steps, which is at most i the number of nodes on the path which leads from the root of the tree to the node. The whole tree is obtained by an endless continuation of the application of the rules to the most recently obtained nodes.

The tree is a spanning tree of the dual graph of the tiling. This graph consists in a juxtaposition of elementary cycles of k nodes which correspond to the k infinigons which share a common vertex of the tiling. The tree is also obtained from the dual graph by deleting edges in order to break all cycles of k nodes. It is enough to delete exactly one edge in such a cycle, in such a way that the remaining graph is connected. Such an edge is called a *missing edge* and the rules also indicate the missing edges which belong only to the dual graph.

The nodes of the tree are the infinigons of the tiling, and the construction of the tree follows the splitting which we described in section 4.1.. A node is attached to each sector which is defined by splitting. This raises no problem when k is even. When k is odd, we have to deal more closely with the half sectors. Indeed, we have to count only once the leading infinigon of a half sector. Below, Figure 8 gives a closer look at the splitting of a half sector. The figure also indicates the order of generation of the considered infinigons.

In order to break each elementary cycle in one point, it is needed to count from 0 up to k. However, if we choose the breaking point in a cycle to fit with possible half sectors, it is enough to count from 0 up to $p{-}1$, where $p = \lfloor \frac{k}{2} \rfloor$ but, in that case, we have to mark the sides. For this reason, we decide to represent each side by a colour.

And so, we associate to each node a colour, red or blue, and a number from 0 up to $p{-}1$. To each node, there is a single rule taken from the six types of rules that are given in Figure 9 for the odd case and Figure 10 for the even case. Each rule indicates the colour and the number that is defined for each of the sons of a node.

Let us look more closely at this process. Starting from $\mathcal{I}$, the leading pentagon of sector $\mathcal{S}$, we may assume that $\mathcal{I}$ is defined by the reflection of an outer infinigon in one of its sides. This side of $\mathcal{I}$ defines the *father side*. Then, we shall orient the infinigon. We assign **0** to the father side and positive numbers, from **1** to infinity to all sides starting from the left neighbour of the father side, anticlockwise. Now we assign negative numbers, from **−1** to infinity to all the other sides, starting from the right neighbour of the father side, clockwise. These numbers are called the *addresses* of the sides of $\mathcal{I}$, and they must not be confused with the indexing of the nodes, from 0 up to $p{-}1$.

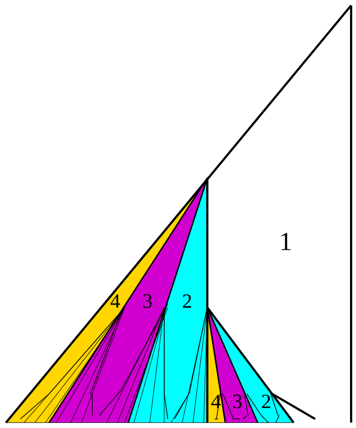

Figure 8 Looking closer at the splitting when k is odd

A reflection in sides n, $n > 1$ or $n < 0$, does not define all the leading infinigons that share a vertex with $\mathcal{I}$: indeed, the vertex shared by sides n and $n{+}1$ is surrounded by k infinigons. We have three of them: $\mathcal{I}$ and its reflected images in sides n and $n{+}1$, call them $\mathcal{I}_n$ and $\mathcal{I}_{n+1}$, respectively. There are $k-3$ other ones. If k is odd, $k = 2p+1$, this indicates that we have to also define $2p-2 = 2(p-1)$ additional sectors.

This is performed by colouring and numbering the nodes: a series of red nodes, with numbers from $p-1$ downto 0 and a series of blue nodes from $p-2$ down to 0. Altogether, both series define $2p-1$ nodes that almost constitute an elementary cycle of the dual graph. In order to establish this point, we have indeed two cases, depending on whether this chain starts from the same node or from two 'neighbouring' nodes.

Consider again two sides with addresses n and $n+1$ and assume that n is positive. From side $\mathbf{-1}$ of $\mathcal{I}_n$ we start a blue chain to the right, towards $\mathcal{I}_{n+1}$, and from side $\mathbf{1}$ of $\mathcal{I}_{n+1}$, we start a red chain to the left, towards $\mathcal{I}_n$. This is indicated by rule (iv). Starting from $\mathcal{I}_{n+1}$, this node being included as it is red, we have p red nodes. Starting after $\mathcal{I}_n$, this node not being included as it is red, there are $p-1$ blue nodes. This gives us $2p-1$ nodes. But $\mathcal{I}_n$ and $\mathcal{I}$ also belong to the cycle and so, we get $2p+1$ nodes, as required. Notice that we have the same situation for negative n when $n < -1$.

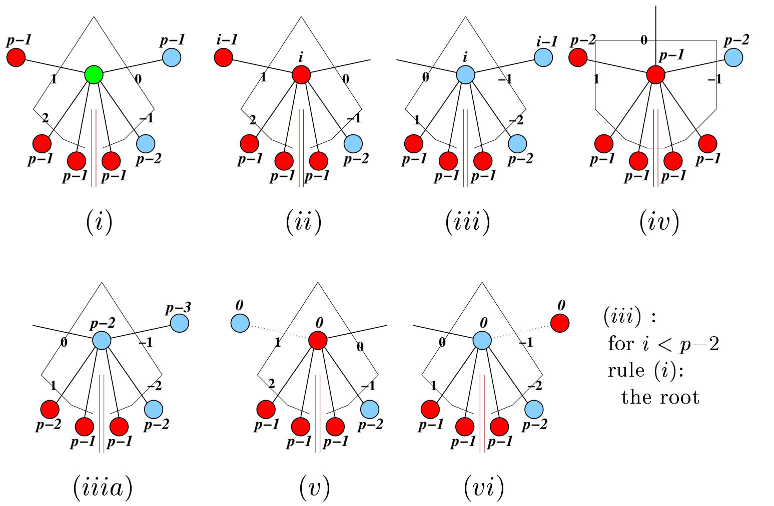

Figure 9 Rules to dispatch the edges, case when k is odd

For positive and negative addresses, the construction of the cycle is controlled by rules (ii) for red nodes and by rules (iii) with rule $(iiia)$ for blue nodes. The need for rule $(iiia)$ appears when we consider the case of sides $\mathbf{-2}$ and $\mathbf{-1}$, when $\mathcal{S}_{-1}$ is associtated to a blue node, while all other $\mathcal{I}_n$'s, with $n \in \mathbb{Z}$ and $n \notin \{-1, 0\}$ are red. When we start the cycle from $\mathcal{I}_{-1}$ towards $\mathcal{I}_{-2}$, as $\mathcal{I}_{-1}$ is represented by a blue node, the computation of the previous paragraph tells us that the red part of the chain must start from $p{-}2$ in order to obtain the correct number of nodes for the elementary cycle of the dual graph.

We have now to deal with another case: that of a vertex shared by side **1** of a leading infinigon $\mathcal{I}$ and side $\mathbf{-1}$ of another infinigon, say $\mathcal{J}$, on the left side of $\mathcal{I}$. In that case, we consider again the chains constructed by application of the rules. From $\mathcal{I}_1$ we have p nodes, $\mathcal{I}_1$ included. From $\mathcal{J}_{-1}$, we have $p{-}1$ blue nodes, $\mathcal{J}_{-1}$ included. We have to add $\mathcal{I}$ and $\mathcal{J}$ to the elementary cycle which we consider and this makes $2p{+}1$ nodes as required.

Notice that rules (v) and (vi) control the completion of the cycle: there is no edge between a (0,*red*) node and a (0,*blue*) one: this corresponds to the edge that is erased from the dual graph in order to obtain a tree. Also notice that half sectors are (0,*blue*) nodes inside a sector and (0,*red*) nodes on the borders. The situation of the borders is natural: the break is between a (0,*blue*) node and a (0,*red*) one. And so, border nodes are of the same colour. An examination of the rules indicates that it must be the red colour.

We may define the addresses of the sides of the initial infinigon by analogy with the other ones: let us decide that the $p{-}2$ blue node has -1 as its address. This fixes all the other addresses.

We are now ready to study the case when k is even. Let us fix $k = 2p$.

It is enough to reduce the maximum number for red nodes by 1 in order to deal with the case when $k = 2p$ for all rules which were introduced for the case when $k = 2p{+}1$. The situations which we described are indeed the same in the case when k is even. The only change is in the total number of nodes. Whence the transformation of the rules that we indicated.

We check that in the general case of neighbouring sides n and $n{+}1$: from $\mathcal{S}_n$, which is associated with a $(p{-}2,$*red*$)$ node, we have a chain of $p{-}1$ blue nodes that goes toward $\mathcal{S}_{n+1}$, and from $\mathcal{S}_{n+1}$ which is also associated with a $(p{-}2,$*red*$)$ node, we have a chain of $p{-}2$ red nodes, $\mathcal{S}_{n+1}$ not being taken into account. This makes $2p{-}3$ nodes to which we have to add the nodes that are associated to $\mathcal{S}_n$, to $\mathcal{S}_{n+1}$ and to $\mathcal{I}$ itself.

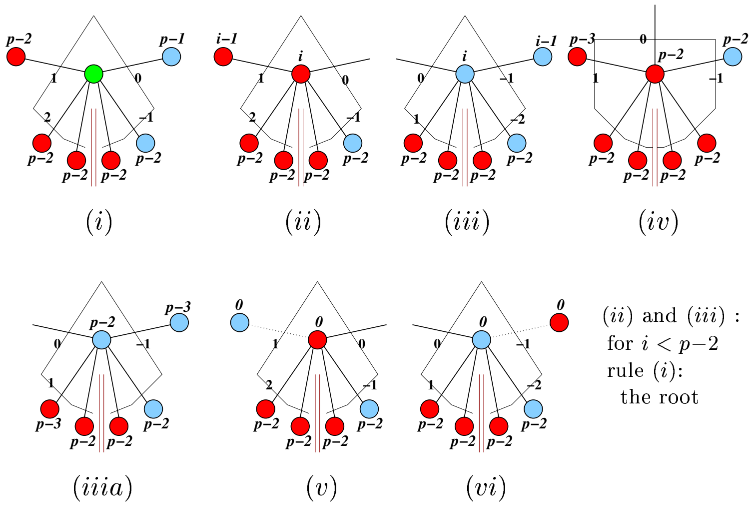

Figure 10 Rules to dispatch the edges, case when k is even

For completeness, we give in Figures 11, 12 and 13 the rules for, respectively, the cases $k=3$, $k=4$ and $k=5$, for which the rules are much simpler. Indeed, we notice that the rules of figures 9 and 10 assume that $p \geq 3$.

Notice that for $k=3$ we have one axiom and a single rule which necessitates a single state. The checking is straightforward.

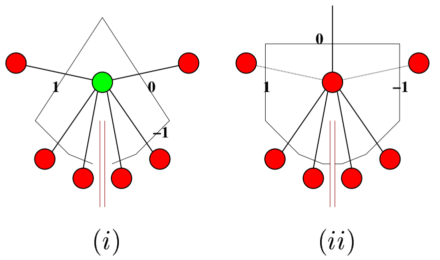

Figure 11 Rules to dispatch the edges, case when $k=3$

For $k=4$, we have two rules besides the axiom and we need three states. The yellow node indicates that in that place there can be either a red node or a blue one, depending on the context. This case is easy to check: it is the case when the interior angles of the infinigon are all right angles.

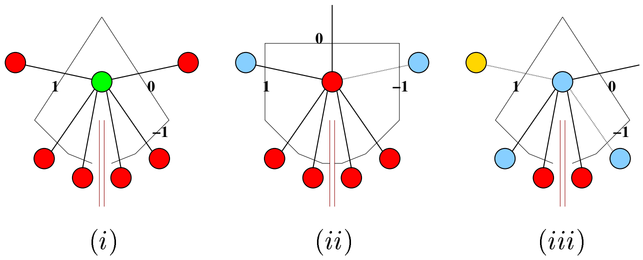

Figure 12 Rules to dispatch the edges, case when $k = 4$

For $k = 5$, we already need three rules besides the axiom and three states which we represent by a blue state and two red ones, 0 and 1.

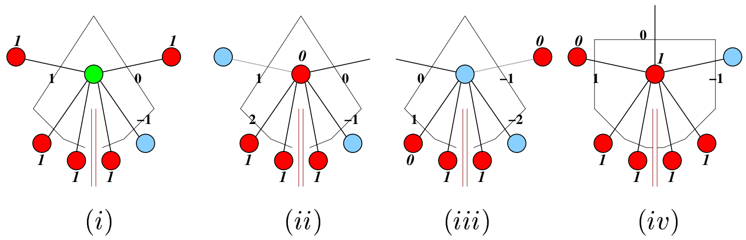

Figure 13 Rules to dispatch the edges, case when $k = 5$

Notice that the same kind of rules can be used for the tiling of the hyperbolic plane by finite regular polygons. This provides us with a combinatorial elementary proof of Poincaré's theorem in the case when one of the angles of the triangle is $\frac{\pi}{2}$.

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