Universality and Cellular Automata

Abstract

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In [14] Wolfram introduces his "Principle of Computational Equivalence" (PCE) which seems to suggest that a certain class of computational processes obeys a 0/1 law: these processes are either decidable or complete. More precisely, unless a process is decidable it is already computationally universal. Thus, Wolfram's principle coexists uneasily with classical recursion theory and in particular the well-developed theory of recursively enumerable degrees, see for example [9]. Since Wolfram relies heavily on discrete dynamical systems such as cellular automata in his approach it is interesting to search for restricted classes of CA and/or alternative notions of universality where a computational classification produces only two classes.

A first formalization of Wolfram's heuristic classification of cellular automata [13] was suggested in [5] where it is shown that the Wolfram-Culik-Yu classes are all undecidable. The classes are located in the arithmetic hierarchy in [10]. In particular it is shown there that testing whether a one-dimensional cellular automaton has only decidable orbits on finite configurations is Σ_3-complete. See also [1] for a critical analysis of frameworks for the classification of cellular automata.

Since cellular automata are naturally discrete dynamical systems rather than models of computation some care has to be taken to avoid awkward input/output conventions. Indeed, it is desirable to find a definition on complexity that avoids coding conventions altogether. In the realm of Turing machines a coding-free approach to universality was suggested by M. Davis in the 1950's, see [3, 4]. More precisely, a Turing machine is *Davis-universal* if the collection of all IDs appearing in finite computations is r.e.-complete. The class of universal Turing machines is strictly contained in the class of Davis-universal machines, and one can recover input/output universality from Davis' concept.

For a cellular automaton ρ it is thus natural to similarly consider the *complete orbit*

$$\mathrm{Orb}_\rho = \{\, (X,Y) \;\big|\; Y = \rho^t(X), t \geq 0 \,\}$$

where all configurations are finite. We then use the Turing degree of Orb_ρ as our measure of complexity of ρ. More precisely, for any r.e. degree $\mathbf{d}$ let

$$\mathbb{C}_{\mathbf{d}} = \{\, \rho \mid \deg(\mathrm{Orb}_\rho) = \mathbf{d} \,\}.$$

Thus, $\mathbb{C}_\emptyset$ is the class of all CA with decidable orbits, corresponding to class *III* in the Wolfram-Culik-Yu hierarchy whereas $\mathbb{C}_{\emptyset'}$ is the class of all computationally universal cellular automata.

In light of the Cook-Wolfram proof of the universality of elementary cellular automaton number 110 one might argue that in general configurations with a finitary description ought to be considered. The universality proof for rule 110 e.g. relies heavily on almost periodic backgrounds: configurations of the form ${}^{\omega}xyz^{\omega}$ are required for the simulation of cyclic tag systems which lies at the heart of the argument. Note that for a fixed rule the existence of undecidable orbits depends on the spatial period of the background configurations under consideration. Indeed, it is undecidable whether a fixed automaton admits undecidable orbits for almost periodic configurations of sufficiently high period, see [11].

At any rate, one can show that $\mathbb{C}_{\mathbf{d}} \neq \emptyset$ for any r.e. degree $\mathbf{d}$, so the classification is non-trivial. As are result, the structure of the upper-semilattice of r.e. degrees is inherited by the orbit-based classification of cellular automata. The latter semilattice is known to be extraordinarily complicated; e.g., its degree is $\emptyset^{(\omega)}$. Membership in $\mathbb{C}_{\mathbf{d}}$ is $\Sigma_3^{\mathbf{d}}$-complete, so that $\mathbb{C}_\emptyset$ is Σ_3-complete whereas $\mathbb{C}_{\emptyset'}$ is Σ_4-complete. This conforms well to one's intuition that it should be more difficult to pinpoint universality than "simplicity". Similar results hold for the analogous cumulative hierarchies $\mathbb{C}_{\leq \mathbf{d}}$ and $\mathbb{C}_{\geq \mathbf{d}}$, see [10, 12].

We show that reversibility is not a suitable restriction in the sense of PCE: every class $\mathbb{C}_{\mathbf{d}}$ contains a reversible CA, see also [6, 2, 7]. Nonetheless, information hiding of sorts (or rather: the lack thereof) seems to be crucial to Wolfram's claim; after all, the standard construction of two incomparable r.e. degrees fails in the sense that the disjoint union of the two sets is complete, see [8].

Changing the basic notion of computational reducibility is also unlikely to produce a simplified hierarchy. Recent work by Simpson has shown that if one adopts Muchnik degrees as the framework there are natural intermediate degrees. Interestingly, one of these natural examples is based on random reals. One should note that at least one elementary cellular automaton, known as rule 30, exhibits strong pseudo-random behavior. It is tempting to speculate that the classification of the orbits of rule 30, on sufficiently general types of configurations, might provide another natural source of intermediate behavior. In particular almost periodic configurations might suffice for this purpose.

References

[1] J. T. Baldwin and S. Shelah. On the classifiability of cellular automata. *Theoretical Computer Science*, 230(1-2):117–129, 2000.

[2] C. H. Bennett. Logical reversibility of computation. *IBM journal of Research and Development*, pages 525–532, 1973.

[3] M. Davis. *A note on universal Turing machines*, pages 167–175. Princeton University Press, 1956.

[4] M. Davis. The definition of universal Turing machines. *Proc. of the American Mathematical Society*, 8:1125–1126, 1957.

[5] K. Culik II and Sheng Yu. Undecidability of CA classification schemes. *Complex Systems*, 2(2):177–190, 1988.

[6] Y. Lecerf. Machine de Turing réversible. Insolubilité récursive en $n \in N$ de l'équation $u = \theta^n u$, où θ est un "isomorphisme de codes". *C. R. Acad. Sci. Paris*, 257:2597–2600, 1963.

[7] K. Morita and M. Harao. Computation universality of 1 dimensional reversible (injective) cellular automata. *Transactions Institute of Electronics, Information and Communication Engineers, E*, 72:758–762, 1989.

[8] R. I. Soare. The Friedberg-Muchnik theorem re-examined. *Canad. J. Math.*, 24:1070–1078, 1972.

[9] R. I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic. Springer Verlag, 1987.

[10] K. Sutner. A note on Culik-Yu classes. *Complex Systems*, 3(1):107–115, 1989.

[11] K. Sutner. Almost periodic configurations on linear cellular automata. *Fundamentae Informaticae*, 58(3,4):223–240, 2003.

[12] K. Sutner. Cellular automata and intermediate degrees. *Theoretical Computer Science*, 296:365–375, 2003.

[13] S. Wolfram. Universality and complexity in cellular automata. *Physica D*, 10:1–35, 1984.

[14] S. Wolfram. *A New Kind of Science*. Wolfram Media, 2002.